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SCHOOL OF HUMANITIES AND SOCIAL SCIENCES

**Mathematical
Methods for Economics**

**MASTER OF ARTS
(ECONOMICS)**

Course Code: ECM-6114

**Course Name: Mathematical Methods for
Economics**



MASTER OF ARTS (ECONOMICS)

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MTS

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— T R I P U R A —

BLOCK-I: INTRODUCTION TO SETS, FUNCTIONS AND NUMBERS

UNIT 1 SETS THEORY

STRUCTURE

- 1.1 Introduction
- 1.2 Objectives
- 1.3 Sets: Definition and examples
- 1.4 Subsets
- 1.5 Some classes of sets
- 1.6 Operations on sets
- 1.7 Laws of operation on sets
- 1.8 Venn diagrams and counting
- 1.9 Summary
- 1.10 Glossary
- 1.11 Exercises
- 1.12 Suggested Readings

1.1 INTRODUCTION

Set theory forms the foundation of modern mathematics. It deals with the concept of collections of objects, known as sets, and the relationships between them. This unit explores the fundamental operations on sets, types of sets, Venn diagrams, and their applications in problem-solving. Understanding set theory is essential for studying advanced topics in mathematics.

1.2 OBJECTIVES

By the end of this unit, you should be able to:

- Define sets and describe elements and membership.

- Explain and apply basic set operations: union, intersection, difference, and complement.
- Illustrate relationships between sets using Venn diagrams.
- Differentiate between types of sets (finite, infinite, null, universal, subset, power set).
- Solve problems involving set identities and laws.
- Apply set theory to real-world contexts and logical reasoning.

1.3 SETS: DEFINITION AND EXAMPLES

Set: A well-defined collection of objects is called a sets.

In our mathematical language, everything in this universe, whether living or nonliving, is called an object.

A given collection of objects is said to be well defined, if we can definitely say whether a given particular object belongs to the collection or not.

The objects in a set are called its members or elements or points. We denote sets by capital letters A, B, C, X, Y, Z , etc.

If a is an element of a set A , we write, $a \in A$, which means that a belongs to A or that a is an element of A .

If a does not belong to A , we write, $a \notin A$.

Examples:

(i) The collection of all vowels in the English alphabet contains five elements, namely a, e, i, o, u. So, this collection is well defined and therefore, it is a set.

(ii) The collection of all odd natural numbers less than 10 contains the numbers 1,3,5,7,9.

So, this collection is well defined and therefore, it is a sets.

(iii) The collection of all prime numbers less than 10 contains the numbers 2,3,5.

So, this collection is well defined and therefore, it is a set.

(iv) All possible roots of the quadratic equation $x^2 - x - 6 = 0$ are -2 and 3. So, the collection of all possible roots of $x^2 - x - 6 = 0$ is well defined and therefore, it is a set.

(v) The collection of all rivers of India, is clearly well defined and therefore, it is a set.

Clearly, river Ganga belongs to this set while river Nile does not belong to it.

(vi) The collection of five most talented teachers of India is not a set, since no rule has been given for deciding whether a given teacher is talented or not.

Cardinality of a set: The cardinality of a set is the number of elements it contains. For example, the cardinality of set $A = \{1, 2, 3\}$ is 3.

A set that doesn't contain any elements in it is known as the empty set, and we represent it with the Greek letter ϕ (phi). For instance, the set of all natural numbers less than 1 is an empty set, represented as ϕ .

There are usually two ways of describing a non-empty set:

(1) Roster method, and (2) Set builder method.

(1) Roster Method: In this method, all the elements of a set are listed within curly braces. For instance, the set of positive divisors of 48 includes the numbers 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48. So, this set can be written as $\{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}$. This method follows two conventions:

Convention 1: The order in which elements are listed doesn't matter.

Convention 2: Each element is written only once; there should be no repetition of elements. For example, the set S containing integers between 11 and 41 would be written as $S = \{11, 12, 13, \dots, 41\}$. It's important not to write $S = \{11, 12, 13, \dots, 41, 12\}$, as this violates Convention 2.

The roster method is also used when listing elements of larger sets. In such cases, we may not list all elements but just enough to provide a sense of what the set contains. For example, the set of integers between 0 and 100 can be expressed as $\{0, 1, 2, \dots, 100\}$, and the set of all integers, Z , can be written as $\{0, +1, -2, \dots\}$.

(2) Set Builder Method: In Set Builder Method, we begin by identifying a defining property, denoted as P that characterizes the elements of the set. This property should be unique to the elements of the set and not shared by any other objects. We then represent the set as:

$\{x \mid x \text{ has property } P\}$ or $\{x : x \text{ has property } P\}$

This can be read as 'the set of all x such that x has property P '. For example, we can express the set of all integers as: $Z = \{x \mid x \text{ is an integer}\}$.

1.4 SUBSETS

Take the sets $A = \{1, 3, 4\}$ and $B = \{1, 4\}$ as an example. In this case, every element of set B is also an element of set A . When this happens, meaning that every element of set B belongs to set A , we call B a subset of A . This is written as $B \subseteq A$. It's evident that for any set A , A is certainly a subset of itself, as every element of A is also in A . So, for any set A , $A \subseteq A$. Additionally, for any set A , the empty set $\emptyset$ is also a subset of A , denoted as $\emptyset \subseteq A$.

It's important to realize that if B is not a subset of A , it implies that there must be at least one element in B that is not part of A . In mathematical terms, we can express this as 'there exists an x in B such that x is not in A '.

We can now assert that sets, A and B , are considered equal, meaning they have exactly the same elements, if and only if A is a subset of B ($A \subseteq B$) and B is a subset of A ($B \subseteq A$).

Example 1. Let $A = \{2,3,5\}$ and $B = \{2,3,5,7,9\}$.

Then, every element of A is an element of B .

$$\therefore A \subseteq B \text{ but } A \neq B$$

Hence, A is a proper subset of set B , i.e., $A \subset B$.

Example 2. Let $A = \{1,2\}$ and $B = \{2,3,5\}$.

Then, $1 \in A$ but $1 \notin B$.

$$\therefore A \not\subseteq B$$

Again, $3 \in B$ but $3 \notin A$.

$$\therefore B \not\subseteq A.$$

Thus, $A \not\subseteq B$ and $B \not\subseteq A$.

SOME RESULTS ON SUBSETS:

- I. Every set is a subset of itself
- II. The empty set is a subset of every set.
- III. The total number of subsets of a set containing n elements is 2^n .

Universal Set: If there are some sets under consideration then there happens to be a set which is a superset of each one of the given sets. Such a set is

known as the universal set for those sets. We shall denote a universal set by U .

Example 3. Let $A = \{1,2,3\}$, $b = \{2,3,4,5\}$ and $C = \{6,7\}$.

If we consider the set $U = \{1,2,3,4,5,6,7\}$ then clearly, U is a superset of each of the given sets.

Hence, U is the universal set.

Example 4. When we discuss sets of lines, triangles or circles in two-dimensional geometry, the plane in which these lines, triangles or circles lie, is the universal set.

INTERVALS AS SUBSETS OF $\mathbb{R}$

Let $a, b \in \mathbb{R}$ and $a < b$. Then, we define:

- (i) Closed Interval $[a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}$.
- (ii) Open Interval (a, b) or $]a, b[= \{x \in \mathbb{R}: a < x < b\}$.
- (iii) Right Half Open Interval $[a, b)$ or $[a, b[= \{x \in \mathbb{R}: a \leq x < b\}$.
- (iv) Left Half Open Interval $(a, b]$ or $]a, b] = \{x \in \mathbb{R}: a < x \leq b\}$.

On the real line, we represent these intervals as shown below:

LENGTH OF AN INTERVAL

The length of each of the intervals $[a, b]$, (a, b) , $[a, b)$ and $(a, b]$ is $(b - a)$.

Examples on intervals

- (i) $[-2, 3] = \{x \in \mathbb{R}: -2 \leq x \leq 3\}$
- (ii) $(-2, 3) = \{x \in \mathbb{R}: -2 < x < 3\}$
- (iii) $[-2, 3) = \{x \in \mathbb{R}: -2 \leq x < 3\}$
- (iv) $(-2, 3] = \{x \in \mathbb{R}: -2 < x \leq 3\}$

Power set: The set of all subsets of a given set A is called the power set of A , denoted by $P(A)$.

$$\text{If } n(A) = m \text{ then } n[P(A)] = 2^m$$

Check Your Progress

1. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$. Find $A \cup B$, $A \cap B$ and $A - B$.

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2. In a class of 60 students, 35 study Economics, 28 study Mathematics and 15 study both. Find the number of students who study at least one of the two subjects.

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3. If $U = \{1, 2, 3, \dots, 10\}$, $A = \{2, 4, 6, 8, 10\}$ and $B = \{1, 2, 3, 4, 5\}$, find $(A \cup B)'$ and $A \cap B$.

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1.5 SOME CLASSES OF SETS

FINITE AND INFINITE SETS

An empty set or a nonempty set in which the process of counting of elements surely comes to an end is called a finite set. A set which is not finite is called an infinite set.

The number of distinct elements contained in a finite set A is denoted by $n(A)$.

Examples of finite sets

(i) Let $A = \{2, 4, 6, 8, 10, 12\}$.

Then, A is clearly a finite set and $n(A) = 6$.

(ii) Let $B =$ set of all letters in the English alphabet. Then, $n(B) = 26$ and therefore, B is finite.

(iii) Let $C = \{x: x \in \mathbb{Z} \text{ and } x^2 - 36 = 0\}$. Then, $C = \{-6, 6\}$, which is clearly a finite set and $n(C) = 2$.

(iv) The set of all persons on earth is a finite set.

(v) The set of all animals on earth is a finite set.

Examples of infinite sets

(i) The set of all points on the arc of a circle is an infinite set.

(ii) The set of all points on a line segment is an infinite set.

(iii) The set of all circles passing through a given point is an infinite set.

(iv) The set of all straight lines parallel to a given line, say the x -axis, is an infinite set.

(v) The set of all positive integral multiples of 5 is an infinite set. Let $\mathbb{Z}^+$ be the set of all positive integers.

Then, $\{5x: x \in \mathbb{Z}^+\} = \{5, 10, 15, 20, 25, \dots\}$ is an infinite set.

(vi) Each of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ is an infinite set.

Note: All infinite sets cannot be described in roster form.

For example, the set R of all real numbers cannot be described in this form, since the elements of this set do not follow a particular pattern.

Equal sets: Two nonempty sets A and B are said to be equal, if they have exactly the same elements and we write, $A = B$.

Otherwise, the sets are unequal and we write, $A \neq B$.

Remarks: (i) The elements of a set may be listed in any order. Thus, $\{1,2,3\} = \{2,1,3\} = \{3,2,1\}$.

(ii) The repetition of elements in a set has no meaning. Thus, $\{1,1,2,2,3\} = \{1,2,3\}$.

1.6 OPERATION ON SETS

Sets can be manipulated through various operations. These operations are essential for understanding abstract algebra, as they provide the foundation for more complex algebraic structures.

(i) Union: The union of A and B consists of all elements in S that belong to either A or B . We denote this union as $A \cup B$. In mathematical terms, $A \cup B$ is defined as follows:

$$A \cup B = \{x \in S \mid (x \in A) \text{ or } (x \in B)\}$$

For example, if $A = \{1,2\}$ and $B = \{4,6,7\}$, then their union, $A \cup B$, is $\{1,2,4,6,7\}$.

Moreover, if $A = \{1,2,3,4\}$ and $B = \{2,4,6,8\}$, then their union, $A \cup B$, becomes $\{1,2,3,4,6,8\}$.

(ii) Intersection: The intersection of two sets, denoted as " $A \cap B$," results in a new set containing elements that are shared by both set A and set B .

To formally define this intersection, we say that $A \cap B$ is the set of elements x in S where x belongs to A and x also belongs to B .

For example, if we have set $P = \{1, 2, 3, 4\}$ and set $Q = \{2, 4, 6, 8\}$, their intersection, $P \cap Q$, is $\{2, 4\}$. Moreover, for any set A , the intersection of A with itself, $A \cap A$, is simply A . Now, let's imagine we have set $A = \{1, 2\}$ and set $B = \{4, 6, 7\}$. In this scenario, A and B share no common elements, so their intersection, $A \cap B$, is an empty set, denoted as φ . When two sets have an empty intersection, we say that they are disjoint, or mutually disjoint. For example, the sets $\{1, 4\}$ and $\{0, 5, 7, 14\}$ are disjoint because they do not share any common elements.

(iii) Complement: The complement of a set A , represented as A' or A or A^c , encompasses all the elements that do not belong to set A but are part of the universal set U . Mathematically, the complement A' or A or A^c can be defined as $A' = \{x \in X \mid x \notin A\}$.

For example, if we have a universal set U , which contains all the integers from 1 to 10, and we define set A as $\{2, 4, 6, 8\}$, then the complement of set A will include all the integers from 1 to 10 that are not in set A . So, A' will be $\{1, 3, 5, 7, 9, 10\}$, containing all the elements that are in U but not in A .

(iv) Difference: The difference of sets A and B , denoted as $A - B$, creates a new set comprising all the elements that are in A but not in B .

Let us consider sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ as an example. The set of all elements in A that are absent in B is $\{1\}$, and we refer to this set as the difference $A - B$. Similarly, the difference $B - A$ comprises the elements in B that are not part of A , which in this case is $\{4\}$. Hence, for any two subsets A and B of a set S , the difference $A - B$ can be defined as the set of elements x in S where x belongs to A but is not part of B :

$$A - B = \{x \in S \mid x \in A \text{ and } x \notin B\}$$

1.7 LAWS OF OPERATIONS ON SETS

1. **Idempotent Laws:** For any set A , (i) $A \cup A = A$ (ii) $A \cap A = A$.
2. **Identity Laws:** For any set A ,
(i) $A \cup \phi = A$ (ii) $A \cap U = A$. Where U is universal set.
3. **Commutative laws:** For any two sets A and B ,
(i) $A \cup B = B \cup A$ [Commutative law for union of sets]
(ii) $A \cap B = B \cap A$ [Commutative law for intersection of sets]
4. **Distributive laws:** For any three sets A, B, C
(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [Distributive law for intersection over union]
(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ [Distributive law for union over intersection]
5. **De Morgan's laws:** For any two sets A and B ,
(i) $(A \cup B)^c = A^c \cap B^c$
(ii) $(A \cap B)^c = A^c \cup B^c$

1.8 VENN DIAGRAMS AND COUNTING

In order to express the relationship among sets in perspective, we represent them pictorially by means of diagrams, called Venn diagrams. In these diagrams, the universal set is represented by a rectangular region and its subsets by circles inside the rectangle. We represent disjoint sets ($A \cap B = \phi$) by disjoint circles and intersecting sets ($A \cap B \neq \phi$) by intersecting circles.

VENN DIAGRAMS IN DIFFERENT SITUATIONS

CASE 1. When the universal set and its subset are given

Let U be the universal set and let $A \subset U$. We draw a circle inside a rectangle. The rectangular region represents U and the circular region represents A .

Example 1. Let $U = \{1,2,3,4,5,6\}$ and $A = \{1,3,5,7\}$.

Then, we draw the Venn Diagram, as shown in the given figure 1.

Clearly $A^c = \{2,4,6\}$.

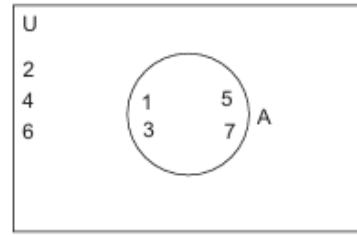


Figure 1.1: Example Ven diagram

CASE 2. When two intersecting subsets of U are given

For representing two intersecting subsets A and B of U , we draw two intersecting circles within the rectangle. The common region of these circles represents $A \cap B$.

Excluding the region of B from that of A shows $(A - B)$.

Excluding the region of A from that of B shows $(B - A)$.

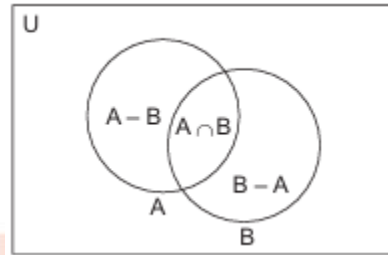


Figure 1.2: Example Ven diagram

Example 2. Let $U = \{1,2,3,4,5,6,7,8\}$ and $A = \{1,3,4,5\}$ and $B = \{2,4,5,6\}$ be its subsets.

Then, we draw the Venn Diagram, as shown in the given figure 3.

Clearly $A \cap B = \{4,5\}$, $A - B = \{1,3\}$, $B - A = \{2,6\}$.

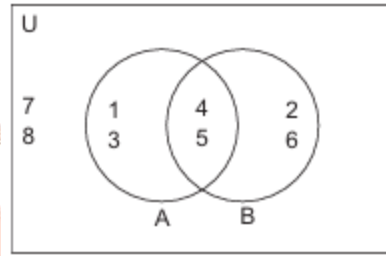


Figure 1.3: Example Ven diagram

CASE 3. When two disjoint subsets of a set be given

In order to represent two disjoint subsets A and B of the universal set U , we draw two disjoint circles within a rectangle.

Example 3. Let $U = \{1,2,3,4,5,6,7\}$ be the universal set, and let $A = \{1,3,5\}$ and $B = \{2,4\}$ be two of its disjoint subsets.

Clearly, $A \cap B = \phi$.

So, we may draw the Venn diagram, as shown in the adjoining figure 4.

Clearly, $A \cap B = \phi$, $(A - B) = \{1,3,5\} = A$ and $(B - A) = \{2,4\} = B$.

$A' = \{2,4,6,7\}$ and $B' = \{1,3,5,6,7\}$.

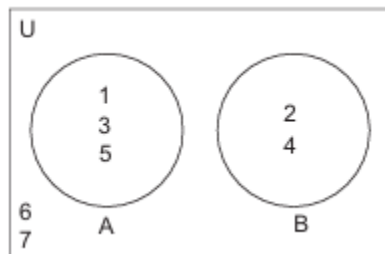


Figure 1.4: Example Ven diagram

CASE 4. When $B \subseteq A \subseteq U$

In this case, we draw two concentric circles within a rectangular region.

The inner circle represents B and the outer circle represents A .

Example 4. Let $U = \{1,2,3,4,5,6,7,8\}$ be the universal set, and let $A = \{1,3,5,7\}$ and $B = \{3,7\}$ be its subsets.

Then, clearly $B \subseteq A$.

Now, we may draw the Venn diagram, as shown in the given figure 5.

$$\begin{aligned} A \cap B &= B = \{3,7\}, A \cup B = A = \{1,3,5,7\} \\ (A - B) &= \{1,5\}, (B - A) = \phi \\ A^c &= \{2,4,6,8\} \text{ and } B' = \{1,5,2,4,6,8\} \end{aligned}$$

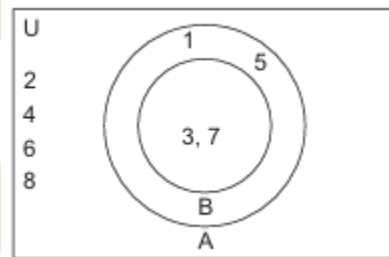
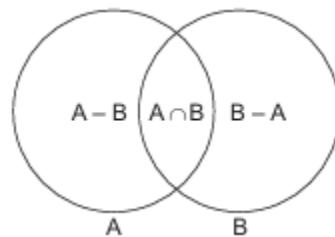


Figure 1.5: Example Ven diagram

IMPORTANT COUNTING RESULTS FROM VENN DIAGRAMS

Let A and B be two intersecting subsets of U .

In counting the elements of $(A \cup B)$, the elements of $A \cap B$ are counted twice, once in counting the elements of A and second time in counting the elements of B .



$$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

If $A \cap B = \phi$, then $n(A \cap B) = 0$ and therefore, in this case, we have

$$n(A \cup B) = n(A) + n(B)$$

From the Venn diagram, it is also clear that

$$(i) n(A - B) + n(A \cap B) = n(A)$$

$$(ii) n(B - A) + n(A \cap B) = n(B)$$

$$(iii) n(A - B) + n(A \cap B) + n(B - A) = n(A \cup B).$$

1.9 SUMMARY

In this unit we have covered the following points:

- Set: A well-defined collection of objects.
- Operations on Sets: Union, intersection, difference, and complement.
- Types of Sets:
 - Finite and infinite sets
 - Empty set (ϕ)
 - Universal set (U)
 - Subset ($A \subseteq B$) and Proper Subset
 - Power set (P(A))
- Venn Diagrams: Visual tools for representing sets and their relationships.
- Set Laws: Commutative, associative, distributive, identity, and De Morgan's laws.

1.10 GLOSSARY

Set: A well-defined collection of distinct objects.

Element: An individual member of a set.

Subset: A set in which every element belongs to another set.

Universal Set: The set containing all objects under consideration.

Empty Set: A set containing no elements.

Union: The set containing all elements belonging to either or both sets.

Intersection: The set containing elements common to two or more sets.

Complement: The set of elements in the universal set that are not in a given set.

1.11 EXERCISES

- 1 If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, verify that:
 - (i) $A \cup B = B \cup A$
 - (ii) $A \cup C = C \cup A$
 - (iii) $B \cup C = C \cup B$
 - (iv) $A \cap B = B \cap A$
 - (v) $B \cap C = C \cap B$
 - (vi) $A \cap C = C \cap A$
 - (vii) $(A \cup B) \cup C = A \cup (B \cup C)$
 - (viii) $(A \cap B) \cap C = A \cap (B \cap C)$
- 2 If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, verify that:
 - (i) $A \cap (B - C) = (A \cap B) - (A \cap C)$
 - (ii) $A - (B \cap C) = (A - B) \cup (A - C)$
- 3 If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$, verify that:
 - (i) $(A \cup B)' = (A' \cap B')$
 - (ii) $(A \cap B)' = (A' \cup B')$
- 4 Let $A = \{a, b, c\}$, $B = \{b, c, d, e\}$ and $C = \{c, d, e, f\}$ be subsets of $U = \{a, b, c, d, e, f\}$. Then, verify that:
 - (i) $(A')' = A$

- (ii) $(A \cup B)' = (A' \cap B')$
 (iii) $(A \cap B)' = (A' \cup B')$
- 5 Given an example of three sets A, B, C such that $A \cap B \neq \phi, B \cap C \neq \phi, A \cap C \neq \phi$ and $A \cap B \cap C = \phi$.
- 6 Let $A = \{a, b, c, e, f\}, B = \{c, d, e, g\}$ and $C = \{b, c, f, g\}$ be subsets of the set $U = \{a, b, c, d, e, f, g, h\}$.
 Draw Venn diagrams to represent the following sets:
 (i) $A \cap B$
 (ii) $A \cup (B \cap C)$
 (iii) $A - B$
 (iv) $B - A$
 (v) $A - (B \cap C)$
 (vi) $(B - C) \cup (C - B)$
- 7 Let $A = \{2, 4, 6, 8, 10\}, B = \{4, 8, 12, 16\}$ and $C = \{6, 12, 18, 24\}$.
 Using Venn diagrams, verify that:
 (i) $(A \cup B) \cup C = A \cup (B \cup C)$
 (ii) $(A \cap B) \cap C = A \cap (B \cap C)$
- 8 Let $A = \{a, e, i, o, u\}, B = \{a, d, e, o, v\}$ and $C = \{e, o, t, m\}$.
 Using Venn diagrams, verify the following:
 (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
 (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1.12 SUGGESTED READINGS

1. Halmos, P. R. (1974). *Naive Set Theory*. Springer.
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3. Pinter, C. C. (2014). *A Book of Set Theory*. Dover Publications.

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UNIT 2 NUMBER SYSTEM

STRUCTURE

- 2.1. Introduction
- 2.2. Objectives
- 2.3. Types of Numbers
- 2.4. Operation on Numbers
- 2.5. Properties of Number Systems
- 2.6. Applications of Number Systems
- 2.7. Some additional topics
- 2.8. Summary
- 2.9. Glossary
- 2.10 Exercises
- 2.11 Suggested Readings

2.1 INTRODUCTION

The number system is a fundamental concept in mathematics, encompassing various ways in which numbers are represented and manipulated. The system plays a crucial role in all areas of mathematics and its applications, from basic arithmetic to advanced mathematical theories. This unit explores the types of number systems, including natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers, and complex numbers. Understanding these different types of numbers and their properties is essential for solving a wide range of mathematical problems and for applications in fields such as engineering, economics, and computer science.

2.2 OBJECTIVES

After going through this unit, students will be able to

- Understand and differentiate between various types of numbers in the number system.
- Perform operations with different types of numbers such as addition, subtraction, multiplication, and division.
- Apply the concepts of rational and irrational numbers in mathematical problem-solving.
- Recognize the significance of the real number system and its properties.
- Understand complex numbers and perform operations involving them.
- Utilize number systems effectively in real-world applications.

2.3 TYPES OF NUMBERS

2.3.1 Natural Numbers

Natural numbers are the basic counting numbers starting from 1 (i.e., 1, 2, 3, 4, 5, ...). They are used for counting and ordering.

2.3.2 Whole Numbers

Whole numbers include all natural numbers along with zero. The set of whole numbers is $\{0, 1, 2, 3, 4, \dots\}$.

2.3.3 Integers

Integers extend whole numbers to include negative numbers. The set of integers is $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

2.3.4 Rational Numbers

Rational numbers are numbers that can be expressed as the quotient of two integers $\frac{p}{q}$, where $q \neq 0$. Examples include $\frac{1}{2}$, $-\frac{3}{4}$, 5.

2.3.5 Irrational Numbers

Irrational numbers cannot be written as a fraction of two integers. They have non-repeating and non-terminating decimal expansions. Examples include $\sqrt{2}$, π , e .

2.3.6 Real Numbers

Real numbers include both rational and irrational numbers. The set of real numbers is denoted by $\mathbb{R}$ and represents all numbers that can be located on the number line.

2.3.7 Complex Numbers

Complex numbers consist of a real part and an imaginary part, typically expressed as $a + bi$, where i is the imaginary unit and a, b are real numbers. The set of complex numbers is denoted by $\mathbb{C}$.

2.4 OPERATION ON NUMBERS

2.4.1 Addition and Subtraction

Addition and subtraction can be performed on all types of numbers, but the results may belong to different subsets of the number system.

- **Natural and Whole Numbers:** Closed under addition and subtraction.
- **Integers:** Closed under addition and subtraction.

- **Rational Numbers:** Closed under addition and subtraction.
- **Irrational Numbers:** The sum or difference of an irrational and a rational number is irrational.

2.4.2 Multiplication and Division

Multiplication and division operations work similarly for different types of numbers.

- **Natural and Whole Numbers:** Closed under multiplication; division may result in fractions or decimals.
- **Integers:** Closed under multiplication; division can result in a rational number.
- **Rational Numbers:** Closed under multiplication and division (except by zero).
- **Irrational Numbers:** Multiplying an irrational number by a rational number results in an irrational number, unless the result is a rational number.

2.5 PROPERTIES OF NUMBER SYSTEMS

2.5.1 Closure Property

The closure property states that performing operations such as addition, subtraction, multiplication, or division on a set of numbers results in a number that is also in the set.

2.5.2 Commutative Property

The commutative property states that the order of numbers in addition or multiplication does not affect the result.

- $a + b = b + a$
- $a \times b = b \times a$

2.5.3 Associative Property

The associative property states that the grouping of numbers does not affect the result.

- $(a + b) + c = a + (b + c)$
- $(a \times b) \times c = a \times (b \times c)$

3.4 Distributive Property

2.5.4 Distributive Property

The distributive property connects multiplication and addition.

- $a \times (b + c) = a \times b + a \times c$

2.6 APPLICATIONS OF NUMBER SYSTEMS

2.6.1 Practical Applications of Natural and Whole Numbers

Used in counting, ordering, and in algorithms such as sorting and searching.

2.6.2 Applications of Rational Numbers

Rational numbers are used in financial calculations, measurements, and statistics.

2.6.3 Applications of Irrational Numbers

Irrational numbers, such as π and e , are used in fields like physics, engineering, and finance.

2.6.4 Applications of Complex Numbers

Complex numbers are used in electrical engineering, control systems, and quantum mechanics, where they model phenomena such as alternating current circuits and waveforms.

Check Your Progress

1. Simplify: $\frac{3}{4} + \frac{5}{6} - \frac{1}{3}$.

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2. Express 0.375 as a rational number in lowest terms and determine whether $\sqrt{2}$ is rational or irrational.

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3. Evaluate: $(\frac{2}{3}) \div (\frac{4}{9}) + \frac{5}{6}$.

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2.7 SOME ADDITIONAL TOPICS

2.7.1 Arithmetic

The first assumption is that the real numbers satisfy all the usual laws of addition, subtraction, multiplication and division.

The rules of arithmetic, of course, include the proviso that division by zero is not allowed. Thus, for example, the expression

$$\frac{2}{0}$$

makes no sense at all. In particular, it is not true that

$$\frac{2}{0} = \infty$$

We shall have a great deal of use for the symbol ∞ , but it must clearly be understood that ∞ does not represent a real number. Nor can it be treated as such except in very special circumstances.

2.7.2 Inequalities

The next assumptions concern inequalities between real numbers and their manipulation.

We assume that, given any two real numbers a and b , there are three mutually exclusive possibilities:

- (i) $a > b$ (a is greater than b)
- (ii) $a = b$ (a equals b)
- (iii) $a < b$ (a is less than b).

Observe that $a < b$ means the same thing as $b > a$. We have, for example, the following inequalities.

$$1 > 0; 3 > 2; 2 < 3; -1 < 0; -3 < -2$$

There is often some confusion about the statements

(iv) $a \geq b$ (a is greater than or equal to b)

(v) $a \leq b$ (a is less than or equal to b).

To clear up this confusion, we note that the following are all true statements.

$$1 \geq 0; 3 \geq 2; 1 \geq 1; 2 \leq 3; -1 \leq 0; -3 \leq -2$$

We assume four basic rules for the manipulation of inequalities. From these, the other rules may be deduced.

(I) If $a > b$ and $b > c$, then $a > c$

(II) If $a > b$ and c is any real number, then

$$a + c > b + c$$

(III) If $a > b$ and $c > 0$, then $ac > bc$ ($c > 0$, inequalities can be multiplied through by a positive factor).

(IV) If $a > b$ and $c < 0$, then $ac < bc$ ($c < 0$, multiplication by a negative factor reverses the inequality).

Note: If x and y are positive, then $x < y$ if and only if $x^2 < y^2$,

2.7.3 Roots

Let n be a natural number. The reader will be familiar with the notation $y = x^n$. For example, $x^2 = x, x$ and $x^3 = x, x, x$.

Our next assumption about the real number system is the following. Given any $y \geq 0$ there is exactly one value of $x \geq 0$ such that

$$y = x^n$$

If $y \geq 0$, the value of $x \geq 0$ which satisfies the equation $y = x^n$ is called the n th root of y and is denoted by

$$x = y^{1/n}$$

When $n = 2$, we also use the notation $\sqrt{y} = y^{1/2}$. Note that, with this convention, it is always true that $\sqrt{y} \geq 0$. If $y > 0$, there are, of course, two numbers whose square is y . The positive one is $\sqrt{y}$ and the negative one is $-\sqrt{y}$. The notation $\pm\sqrt{y}$ means $\sqrt{y}$ or $-\sqrt{y}$.

If $r = m/n$ is a positive rational number and $y \geq 0$, we define

$$y^r = (y^m)^{1/n}$$

If r is a negative rational, then $-r$ is a positive rational and hence y^{-r} is defined. If $y > 0$ we can therefore define y^r by

$$y^r = \frac{1}{y^{-r}}$$

We also write $y^0 = 1$. With these conventions it follows that, if $y > 0$, then y^r is defined for all rational numbers r .

2.7.4 Quadratic equations

If $y > 0$, the equation $x^2 = y$ has two solutions. We denote the positive solution by $\sqrt{y}$. The negative solution is therefore $-\sqrt{y}$. We note again that there is no ambiguity about these symbols and that $\pm\sqrt{y}$ simply means $\sqrt{y}$ or $-\sqrt{y}$.

The general quadratic equation has the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$. Multiply through by $4a$. We obtain

$$\begin{aligned}4a^2x^2 + 4abx + 4ac &= 0 \\(2ax + b)^2 - b^2 + 4ac &= 0 \\(2ax + b)^2 &= b^2 - 4ac\end{aligned}$$

It follows that the quadratic equation has no real solutions if $b^2 - 4ac < 0$, one real solution if $b^2 - 4ac = 0$ and two real solutions if $b^2 - 4ac > 0$. If $b^2 - 4ac > 0$,

$$\begin{aligned}2ax + b &= \pm\sqrt{(b^2 - 4ac)} \\x &= \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}\end{aligned}$$

The roots of the equation $ax^2 + bx + c = 0$ are therefore

$$\alpha = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a} \text{ and } \beta = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}$$

It is a simple matter to check that, for all values of x ,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

With the help of this formula, we can sketch the graph of the equation $y = ax^2 + bx + c$.

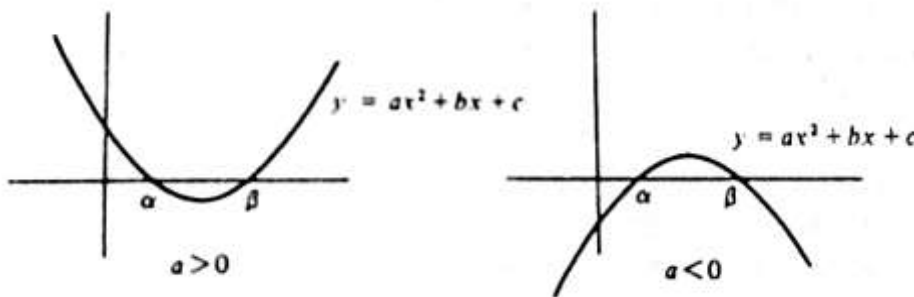


Figure 2.1: Graph of $y = ax^2 + bx + c$

2.8 SUMMARY

In this unit we have covered the following topics:

- The number system consists of different types of numbers: natural, whole, integers, rational, irrational, real, and complex numbers.
- Each set has its own properties and operations, and understanding these is crucial for solving mathematical problems.
- The number system is applied in various real-world contexts, including finance, engineering, and physics, making it a foundational concept in mathematics and its applications.

2.9 GLOSSARY

Natural Numbers: Positive counting numbers beginning with 1.

Whole Numbers: Natural numbers together with zero.

Integer: A positive or negative whole number, including zero.

Rational Number: A number expressible as the ratio of two integers.

Irrational Number: A number that cannot be expressed as a ratio of two integers.

Real Number: A number that can be represented on the real number line.

Prime Number: A positive integer greater than one having exactly two positive factors.

Composite Number: A positive integer greater than one having more than two factors.

2.10 EXERCISES

- 1 Classify the following numbers as natural, whole, integer, rational, or irrational:
 - a) $\frac{3}{4}$
 - b) -5
 - c) π
 - d) 0.75
 - e) $\sqrt{3}$
- 2 Perform the following operations and state the result's type of number:
 - a) $7 + (-4)$
 - b) $\frac{5}{2} \times 3$
 - c) $\frac{2}{3} - \frac{1}{4}$
 - d) $\sqrt{5} \times \pi$
- 3 Simplify the following expressions and classify the results:
 - a) $3 + (-7)$
 - b) $\sqrt{9} - 2$
 - c) $2 \times \sqrt{2}$
- 4 Solve the following real-world problem:

The height of a water tank is given by the formula $h = 5t + 2$, where h is the height (in meters) and t is the time (in hours). After

4 hours, calculate the height of the water tank. Is the result a natural number, whole number, integer, rational, or irrational?

2.11 SUGGESTED READINGS

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UNIT 3 FUNCTIONS AND THEIR GRAPHS

STRUCTURE

- 3.1. Introduction
- 3.2. Objectives
- 3.3. Functions
- 3.4. Types of functions
- 3.5. Inverse function and composition of functions
- 3.6. Graphs of some basic functions
- 3.7. Solved examples
- 3.8. Summary
- 3.9. Glossary
- 3.10 Exercises
- 3.11 Suggested Readings

3.1 INTRODUCTION

Functions are fundamental in mathematics, providing a way to describe relationships between variables. This unit explores the concept of functions, their various types, inverse functions, and the composition of functions. Additionally, the graphical representation of functions will be covered to enhance understanding of their behavior and properties. Mastering functions and their graphs is crucial for studying calculus, algebra, and applied mathematics.

3.2 OBJECTIVES

After going through this unit, students will be able to

- Define functions and explain the concept of domain, range.
- Identify and describe different types of functions.

- Understand and compute inverse functions and perform function composition.
- Graph basic functions and interpret their key features.

3.3 FUNCTIONS

Functions are mathematical tools that describe the relationship between elements of two sets. Given two non-empty sets, A and B , a function f from A to B is essentially a rule that assigns precisely one element from B to each element of A . In this context, when we say $f(a) = b$, it signifies that ' b ' is the unique element in B that the function f associates with the element ' a ' in A . We formally represent this as $f: A \rightarrow B$. Functions are sometimes referred to as mappings or transformations.

3.3.1 Domain and Range of a function:

In the context of a function from A to B , we designate A as the domain of the function, representing the set from which input elements are drawn. B is identified as the co-domain of the function, signifying the encompassing set to which the output elements belong. When we state that $f(a) = b$, we essentially express that ' b ' is the image of ' a ,' and ' a ' is considered a pre-image of ' b '. The range, or image, of the function comprises all the images of elements from the domain A . Also, it's appropriate to say that the function f maps elements from A to B .

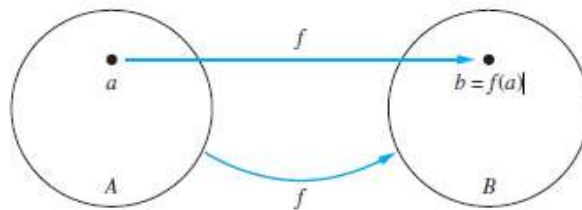


Figure 3.1: Function f from A to B

Example: We have a function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ that assigns the square of an integer to the integer itself. In other words, for any integer x , $f(x) = x^2$. Here, the domain of f encompasses all integers, the co-domain of f includes all integers, and the range of f specifically comprises integers that are perfect squares, which are $\{0, 1, 4, 9, \dots\}$.

It is to be noted that if we have two functions, f_1 and f_2 , both mapping elements from set A to the real numbers R , we can define the functions $f_1 + f_2$ and $f_1 f_2$ as follows for all x in A :

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) * f_2(x)$$

3.3.2 Image of a subset

If we have a function f from A to B and a subset S of A , the image of S under the function f is the subset of B that comprises the images of the elements within S . We denote this image as $f(S)$, expressed as $f(S) = \{t \mid s \in S, t = f(s)\}$. Alternatively, we can use the shorthand $\{f(s) \mid s \in S\}$ to represent this set.

Please remember that the notation $f(S)$ for the image of the set S under the function f denotes a set and not the value of the function f for the set S .

For instance, consider sets $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$, along with the function $f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1$, and $f(e) = 1$. If we take the subset $S = \{b, c, d\}$, the image of this subset is represented as $f(S) = \{1, 4\}$.

3.4 TYPES OF FUNCTIONS

(i) One-to-One Function (Injective Function): A function f is termed one-to-one, or injective, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f . This means that for each pair of distinct elements a and b in the domain, the function assigns them distinct values. Mathematically, a function is considered one-to-one if, for all elements a and b in the domain, $f(a) = f(b)$ implies $a = b$.

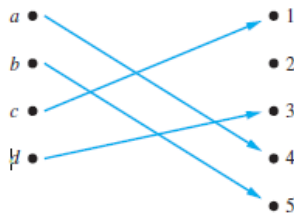


Figure 3.2: One-to-One function

Examples:

1. Consider the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$. The function f is one-to-one because it assigns different values to each element in its domain.
2. Consider the function $f(x) = x^2$ from the set of integers to the set of integers is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$.

(ii) Onto Function (Surjective Function): A function f from A to B is said to be as onto, or surjective, if and only if for every element b in the co-domain B , there exists an element a in the domain A such that $f(a) = b$. In simpler terms, onto function ensures that every element in the co-domain is mapped to by at least one element from the domain. Mathematically, a

function is surjective if $\forall y \exists x (f(x) = y)$, where x ranges over the domain of the function and y over the co-domain.

Examples:

1. Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ with $f(a) = 3, f(b) = 2, f(c) = 1$, and $f(d) = 3$. In this case, f is onto because every element in the co-domain $\{1, 2, 3\}$ has at least one element in the domain $\{a, b, c, d\}$ that maps to it.
2. The function $f(x) = x^2$ from the set of integers to the set of integers is not onto since there is no integer x such that $x^2 = -1$, and hence, not all elements in the co-domain are reached.

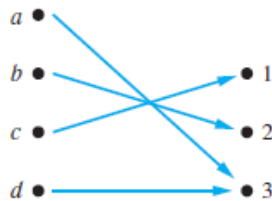


Figure 3.3: Onto function

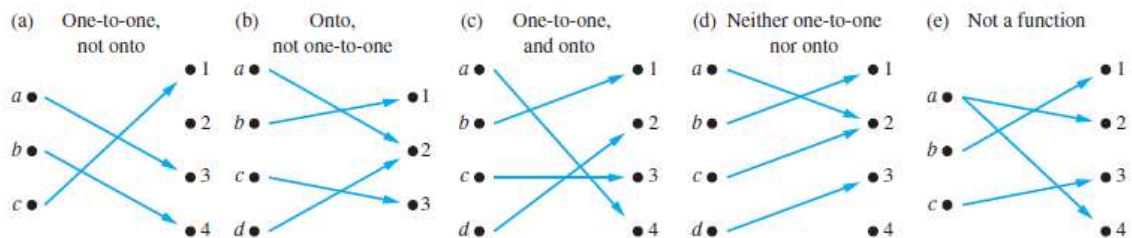


Figure 3.4: Different types of correspondences

(iii) One-to-One Onto Function (Bijective Function): A function f is referred to as a one-to-one onto correspondence, or a bijection, if it possesses both one-to-one and onto properties. We can also say that such a function is bijective.

Example: Consider the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4, f(b) = 2, f(c) = 1$, and $f(d) = 3$. Is f a bijection? Yes, this function is a bijection because it is both one-to-one (no two-domain values map to the same co-domain value) and onto (covers all co-domain elements).

When dealing with functions whose domain and co-domain are subsets of the real numbers, we can classify them as either **increasing or decreasing function**.

(a) Increasing Function: A function f is considered increasing if, for all x and y in its domain, where $x < y$, it holds that $f(x) \leq f(y)$. In simple terms, as the input increases, the function value does not decrease.

(b) Strictly Increasing Function: A function is strictly increasing if, for all x and y in its domain, where $x < y$, it holds that $f(x) < f(y)$. Here, the function values strictly increase as the input increases.

(c) Decreasing Function: A function f is termed decreasing if, for all x and y in its domain, where $x < y$, it holds that $f(x) \geq f(y)$. In this case, as the input increases, the function value does not increase.

(d) Strictly Decreasing Function: A function is strictly decreasing if, for all x and y in its domain, where $x < y$, it holds that $f(x) > f(y)$. In this situation, the function values strictly decrease as the input increases.

3.5 INVERSE FUNCTION AND COMPOSITIONS OF FUNCTIONS

(i) Inverse Function: When we have a one-to-one correspondence (bijective function) from set A to set B , the inverse function of this correspondence assigns to an element b in B the unique element a in A such

that $f(a) = b$. This inverse function is denoted by f^{-1} . Thus, $f^{-1}(b) = a$ when $f(a) = b$.

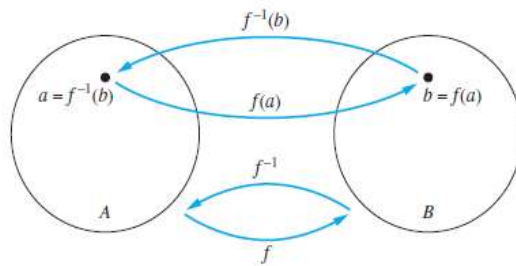


Figure 3.5: Inverse of function $f(x)$

Examples:

1. Consider the function f from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it's a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

2. Let $f: Z \rightarrow Z$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it's a one-to-one correspondence. To reverse the correspondence, suppose y is the image of x , so $y = x + 1$. Then $x = y - 1$. This means that $f^{-1}(y) = y - 1$.

(ii) Composition of Functions: If you have a function g from set A to set B and a function f from set B to set C , the composition of the functions f and g , denoted as $f \circ g$, is defined for all a in set A by $(f \circ g)(a) = f(g(a))$. It's essentially applying the function g first and then the function f .

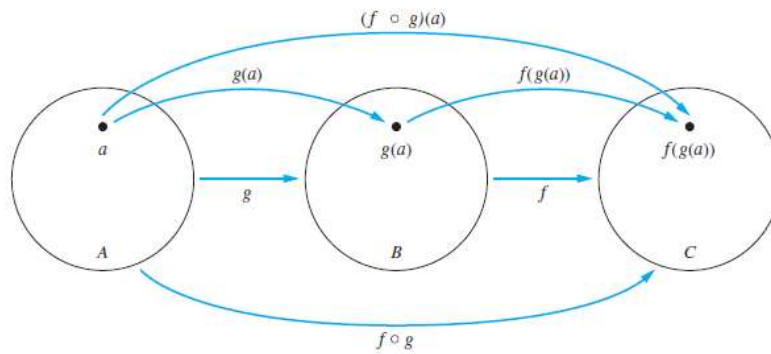


Figure 3.6: Composition of functions $f(x)$ and $g(x)$

Examples:

1. Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b, g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3, f(b) = 2$, and $f(c) = 1$. What is the composition of f and g , and what is the composition of g and f ?

Solution: The composition $f \circ g$ is defined as $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$. Note that $g \circ f$ is not defined because the range of f is not a subset of the domain of g .

2. Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g ? What is the composition of g and f ?

Solution: Both compositions, $f \circ g$ and $g \circ f$, are defined. Moreover, $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$ and $(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Check Your Progress

1. For $f(x) = 2x + 3$, find $f(-2)$, $f(0)$ and $f(5)$.

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2. Find the domain and range of $f(x) = \sqrt{x - 2}$.

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3. Determine whether $f(x) = x^2$ is one-one on $\mathbb{R}$. Give a mathematical reason using suitable values of x .

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3.6 GRAPHS OF SOME BASIC FUNCTIONS

3.6.1 Polynomial functions: A function of the form:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where $n \in \mathbb{N}$ and $a_0, a_1, a_2, \dots, a_n \in \mathbb{R}$.

Then, f is called a polynomial function. " $f(x)$ is also called polynomial in x ".

Some of basic polynomial functions are

(i) Identity function/Graph of $f(x) = x$

A function f defined by $f(x) = x$ for all $x \in \mathbb{R}$, is called the identity function.

Here, $y = x$ clearly represents a straight line passing through the origin and inclined at an angle of 45° with x -axis shown as figure 3.6.1.

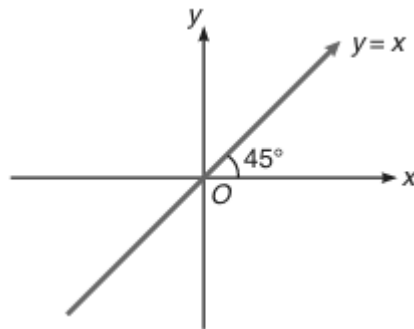


Figure 3.6.1 Graph of $f(x) = x$

The domain and range of identity functions are both equal to R .

(ii) Graph of $f(x) = x^2$

A function given by $f(x) = x^2$ is called the square function.

The domain of square function is R and its range is $R^+ \cup \{0\}$ or $[0, \infty)$

Clearly $y = x^2$, is a parabola. Since $y = x^2$ is an even function, so its graph is symmetrical about y -axis, shown as figure 3.6.2

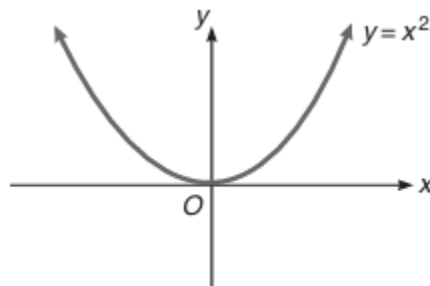


Figure 3.6.2 Graph of $f(x) = x^2$

(iii) Graph of $f(x) = x^3$

A function given by $f(x) = x^3$ is called the cube function.

The domain and range of cube are both equal to R .

Since, $y = x^3$ is an odd function, so its graph is symmetrical about opposite quadrant, i.e., "origin", shown as figure 3.6.3

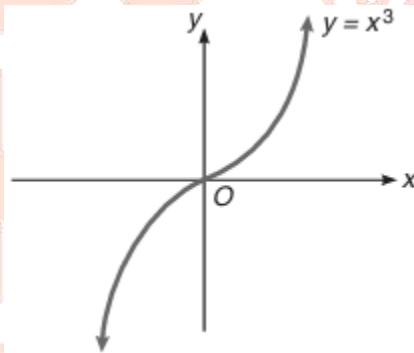


Figure 3.6.3

3.6.2 Rational Expression: A function obtained by dividing a polynomial by another polynomial is called a rational function.

$$\Rightarrow f(x) = \frac{P(x)}{Q(x)}$$

Domain $\in R - \{x \mid Q(x) = 0\}$

i.e., domain $\in R$ except those points for which denominator = 0.

Graphs of some Simple Rational Functions

(i) Graph of $f(x) = \frac{1}{x}$

A function defined by $f(x) = \frac{1}{x}$ is called the reciprocal function or rectangular hyperbola, with coordinate axis as asymptotes. The domain and range of $f(x) = \frac{1}{x}$ is $R - \{0\}$.

Since, $f(x)$ is odd function, so its graph is symmetrical about opposite quadrants. Also, we observe

$$\lim_{x \rightarrow 0^+} f(x) = +\infty \text{ and } \lim_{x \rightarrow 0^-} f(x) = -\infty$$

and

$$\text{as } x \rightarrow \pm\infty \Rightarrow f(x) \rightarrow 0$$

Thus, $f(x) = \frac{1}{x}$ could be shown as figure 3.6.4

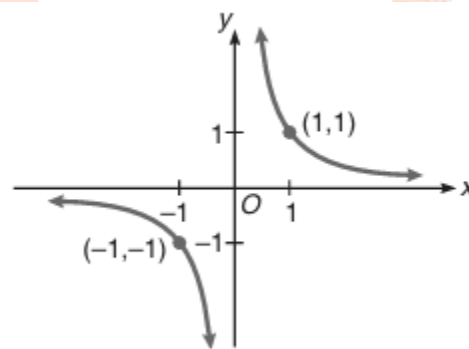


Figure 3.6.4 Graph of $f(x) = \frac{1}{x}$

(ii) Graph of $f(x) = \frac{1}{x^2}$

Here, $f(x) = \frac{1}{x^2}$ is an even function, so its graph is symmetrical about y-axis.

Domain of $f(x)$ is $R - \{0\}$ and range is $(0, \infty)$.

Also, as $y \rightarrow \infty$ as $\lim_{x \rightarrow 0^+} f(x)$ or $\lim_{x \rightarrow 0^-} f(x)$. and $y \rightarrow 0$ as $\lim_{x \rightarrow \pm\infty} f(x)$.

Thus, $f(x) = \frac{1}{x^2}$ could be shown as figure 3.6.5

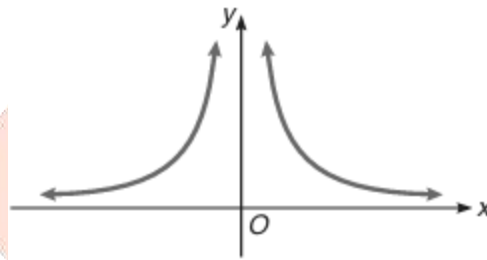


Figure 3.6.5 Graph of $f(x) = \frac{1}{x^2}$

3.6.3 Piecewise Functions

(a) Absolute value function (or modulus function)

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

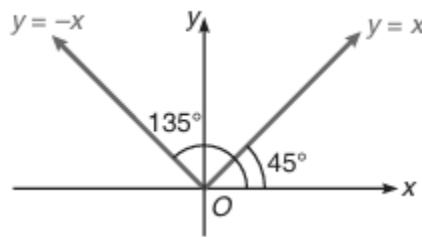


Figure 3.6.6 Graph of $f(x) = |x|$

"It is the numerical value of x ".

"It is symmetric about y -axis" where domain $\in \mathbb{R}$
and

$$\text{range} \in [0, \infty)$$

Properties of modulus functions

- (i) $|x| \leq a \Rightarrow -a \leq x \leq a; (a \geq 0)$
- (ii) $|x| \geq a \Rightarrow x \leq -a \text{ or } x \geq a; (a \geq 0)$
- (iii) $|x \pm y| \leq |x| + |y|$
- (iv) $|x \pm y| \geq ||x| - |y||$.

3.7 SOLVED EXAMPLES

Example 1: Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a function defined by

$$f(x) = \begin{cases} 3x - 1, & \text{when } x > 3 \\ x^2 - 2, & \text{when } -2 \leq x \leq 3 \\ 2x + 3, & \text{when } x < -2 \end{cases}$$

Find (i) $f(2)$, (ii) $f(4)$, (iii) $f(-1)$, (iv) $f(-3)$.

Solution: (i) $f(2) = (2)^2 - 2 = 4 - 2 = 2$

(ii) $f(4) = 3(4) - 1 = 12 - 1 = 11$

(iii) $f(-1) = (-1)^2 - 2 = 1 - 2 = -1$

(iv) $f(-3) = 2(-3) + 3 = -6 + 3 = -3$

Example 2: For the function $y = +\sqrt{x}$, say whether it is a function or not, If it is function, find its domain and range.

Solution: Here, we have $y = +\sqrt{x}$

Since y is real if $x \geq 0$ and is unique and finite for each $x \geq 0$

Then $y = +\sqrt{x}$ is a function with domain $[0, \infty]$

Again, from (1),

$$y \geq 0 \forall x \geq 0$$

Hence,

$$\text{Range} = [0, \infty).$$

Example 3: Find the domain of $f(x) = \frac{x^3 - x^2 + 4x + 7}{3x + 11}$

Solution: Since f is defined for all real values of x except when $3x + 11 = 0$.

i.e., when,

$$x = \frac{-11}{3}$$

Hence, domain of $f = \mathbf{R} - \left\{\frac{-11}{3}\right\}$

Example 4: Let $f: N - \{1\} \rightarrow \mathbf{N}$ be defined by $f(n)$ = the highest prime factor of n . Show that f is neither one-one nor onto. Also, find the range off.

Solution: Since, we have

$$\begin{aligned} f(6) &= \text{the highest prime factor of } 6 = 3 \\ f(9) &= \text{the highest prime factor of } 9 = 3 \\ f(12) &= \text{the highest prime factor of } 12 = 3 \end{aligned}$$

Therefore, f is a many-one function.

Clearly, image of any $n \in N - \{1\}$ is the largest prime number that divides n . So the range of f consists of prime number only. Consequently, range of $f \neq N$ (Co-domain)
 $\Rightarrow f$ is not onto function.

Hence, f is neither one-one nor onto. The range of f is the set of all prime numbers.

Example 5: Find the domain of the function $f(x) = \frac{x^2+3x+1}{x^2-5x+6}$

Solution: Obviously $f(x)$ is not defined when $x^2 - 5x + 6 = 0$

$$\begin{aligned} \text{Now} \quad & (x^2 - 5x + 6) \\ \Rightarrow & 0 \\ \Rightarrow & (x - 2)(x - 3) = 0 \\ \Rightarrow & x = 2, \text{ or } x = 3. \end{aligned}$$

So $f(x)$ is not defined at $x = 2, 3$. Hence domain $f = R - \{2, 3\}$.

Example 6: Let $A = \{1, 2\}$. Find all one-to-one functions from A to A .

Solution: Let $f: A \rightarrow A$ be a one-one function. Then for $f(1)$, there are two choices i.e., 1 or 2.

Let us first suppose $f(1) = 1$

As $f: A \rightarrow A$ is one-one

So, $f(2) = 2$

Therefore, we have $f(1) = 1, f(2) = 2$.

Now, let $f(1) = 2$

Since, $f: A \rightarrow A$ is one-one, therefore $f(2) = 1$

Therefore, we have $f(1) = 2$ and $f(2) = 1$

Hence, we have two one-one functions say f and g from A to A given by
and

$$\begin{aligned} f(1) &= 1, f(2) = 2 \\ f(2) &= 1 \text{ and } f(1) = 2 \end{aligned}$$

3.8 SUMMARY

In this unit we have covered the following points:

- The definition of a function
- The 1-1 function, onto function and bijective function.
- The composition of functions.
- Inverse function and composition of functions
- Graphs of some basic functions

3.9 GLOSSARY

Function: A relation assigning exactly one output to each input.

Domain: The set of all permitted input values of a function.

Range: The set of output values produced by a function.

One-One Function: A function in which distinct inputs have distinct outputs.

Many-One Function: A function in which two or more inputs may have the same output.

Onto Function: A function whose range equals its codomain.

Inverse Function: A function that reverses the input-output correspondence of a function.

Graph of a Function: A set of ordered pairs representing a function on a coordinate plane.

3.10 EXERCISES

1. Let $A = \{-2, -1, 0, 2, 1, 2\}$ and $f: A \rightarrow Z$ given by $f(x) = x^2 - 2x - 3$

Find (i) the range of $f(i)$ pre-image of 6, -3 and 5.

2. Find the domain and range of the function $f(x) = \sqrt{(x-1)(3-x)}$

3. Find the range of the function $f(x) = \frac{1}{(2x-3)(x+1)}$

4. Find the domain and range of the following functions.

(i) $f(x) = \frac{x^2-1}{x-1}$

(ii) $|x| + y = 0$

(iii) $f(x) = \frac{|x-1|}{x-1}$

(iv) $y = \sqrt{x-3}$

(v) $f(x) = x^2 + 4$

5. If $A = \{-1, 0, 2, 5, 6, 11\}$, $B = \{-2, 0, 18, 28, 108\}$ and $f(x) = x^2 - x - 2$, find $f(A)$.

6. Let A be the set of two positive integers. Let $f: A \rightarrow Z^+$, set of positive integers be defined by $f(n) = p$, where p is the highest prime factor of n . If range of $f = \{3\}$, find A .

7. Find the domain for which the function $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.

8. Let $f_1: R \rightarrow R$ and $f_2: C \rightarrow C$ be two functions defined as $f_1(x) = x^3$ and $f_2(x) = x^3$. Show that they are not equal.

3.11 SUGGESTED READINGS

1. Gelfand, I. M., Glagoleva, E. G., & Shnol, E. E. (2002). *Functions and Graphs*. Dover Publications.
2. Agarwal, A. M. (2016). *Play with Graphs*. Arihant Publications.
3. Larson, R., & Hostetler, R. P. (2010). *Precalculus with Limits: A Graphing Approach*. Brooks Cole.
4. Open Textbook Library. (2020). *Modeling, Functions, and Graphs*. University of Minnesota.
5. Stewart, J. (2015). *Precalculus: Mathematics for Calculus*. Cengage Learning.



UNIT 4 PRINCIPLE OF MATHEMATICAL INDUCTION

STRUCTURE

- 4.1. Introduction
- 4.2. Objectives
- 4.3. Principle of Mathematical Induction
- 4.4. Summary
- 4.5. Glossary
- 4.6. Exercises
- 4.7. Suggested Readings

4.1 INTRODUCTION

The Principle of Mathematical Induction is a powerful and widely-used technique in mathematics, particularly in the study of sequences, series, and discrete structures. It provides a systematic method for proving statements that hold for infinitely many cases, typically involving natural numbers.

The principle is based on a simple yet profound idea: if we can show that a statement holds for a base case, and then demonstrate that if the statement holds for any particular case, it also holds for the next case, then we can conclude that the statement holds for all cases beyond the base case.

4.2 OBJECTIVES

After going through this unit, students will be able to

- Understand the Principle of mathematical induction
- Apply the Principle of mathematical induction to solve various numerical problems

4.3 PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true and

(ii) $P(k + 1)$ is true, whenever $P(k)$ is true

then $P(n)$ is true for all $n \in N$.

Example 1: Using the principle of mathematical induction, prove that

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2 \text{ for all } n \in N.$$

Solution: Let the given statement be $P(n)$. Then,

$$P(n): 1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2.$$

Putting $n = 1$ in the given statement, we get

$$\begin{aligned} \text{LHS} &= 1 \text{ and RHS} = 1^2 = 1. \\ \therefore \text{LHS} &= \text{RHS}. \end{aligned}$$

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): 1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2. \quad (1)$$

Now, $1 + 3 + 5 + 7 + \cdots + (2k - 1) + \{2(k + 1) - 1\}$

$$\begin{aligned} &= \{1 + 3 + 5 + 7 + \cdots + (2k - 1)\} + (2k + 1) = k^2 + (2k + 1) \\ &= (k + 1)^2. \text{ [using (1)]} \end{aligned}$$

$$\therefore P(k + 1): 1 + 3 + 5 + 7 + \cdots + (2k - 1) + \{2(k + 1) - 1\} = (k + 1)^2.$$

This shows that $P(k + 1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2 \text{ for all } n \in N.$$

Example 2: Using the principle of mathematical induction, prove that

$$1 + 4 + 7 + 10 + \cdots + (3n - 2) = \frac{1}{2}n(3n - 1) \text{ for all } n \in N.$$

Solution: Let the given statement be $P(n)$. Then,

$$P(n): 1 + 4 + 7 + 10 + \cdots + (3n - 2) = \frac{1}{2}n(3n - 1).$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS} = 1 \text{ and RHS} = \frac{1}{2} \times 1 \times (3 \times 1 - 1) = 1.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): 1 + 4 + 7 + 10 + \cdots + (3k - 2) = \frac{1}{2}k(3k - 1). \quad (1)$$

Now, $1 + 4 + 7 + \cdots + (3k - 2) + \{3(k + 1) - 2\}$

$$= \{1 + 4 + 7 + \cdots (3k - 2)\} + (3k + 1)$$

$$= \frac{1}{2}k(3k - 1) + (3k + 1) \quad [\text{using (1)}]$$

$$= \frac{1}{2}(3k^2 - k + 6k + 2) = \frac{1}{2}(3k^2 + 5k + 2)$$

$$= \frac{1}{2}(3k^2 + 3k + 2k + 2) = \frac{1}{2} \cdot \{3k(k + 1) + 2(k + 1)\}$$

$$= \frac{1}{2}(k + 1)(3k + 2) = \frac{1}{2}(k + 1)\{3(k + 1) - 1\}.$$

$$\begin{aligned} \therefore P(k + 1): 1 + 4 + 7 + \cdots + \{3(k + 1) - 2\} \\ = \frac{1}{2}(k + 1)\{3(k + 1) - 1\}. \end{aligned}$$

This shows that $P(k + 1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{1}{2}n(3n - 1) \text{ for all } n \in N.$$

Example 3: Using the principle of mathematical induction, prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1) \text{ for all } n \in N.$$

Solution: Let the given statement be $P(n)$. Then,

$$P(n): 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1).$$

Putting $n = 1$ in the given statement, we get

$$\begin{aligned} \text{LHS} &= 1^2 = 1 \text{ and RHS} = \frac{1}{6} \times 1 \times 2 \times (2 \times 1 + 1) = 1. \\ \therefore \text{LHS} &= \text{RHS}. \end{aligned}$$

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{1}{6}k(k + 1)(2k + 1).$$

Now, $1^2 + 2^2 + 3^2 + \cdots + k^2 + (k + 1)^2$

$$\begin{aligned} &= \{1^2 + 2^2 + 3^2 + \cdots + k^2\} + (k + 1)^2 \\ &= \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2 \end{aligned}$$

[using (i)]

$$\begin{aligned} &= \frac{1}{6}(k + 1) \cdot \{k(2k + 1) + 6(k + 1)\} = \frac{1}{6}(k + 1)(2k^2 + 7k + 6) \\ &= \frac{1}{6}(k + 1)\{(2k^2 + 4k) + (3k + 6)\} \end{aligned}$$

$$= \frac{1}{6}(k + 1)(k + 2)(2k + 3) = \frac{1}{6}(k + 1)(k + 1 + 1)[2(k + 1) + 1].$$

$$\begin{aligned} \therefore P(k + 1): 1^2 + 2^2 + 3^2 + \cdots + (k + 1)^2 \\ = \frac{1}{6}(k + 1)(k + 1 + 1)[2(k + 1) + 1]. \end{aligned}$$

This shows that $P(k + 1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1) \text{ for all } n \in N.$$

Example 4: Using the principle of mathematical induction, prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2 \text{ for all } n \in N.$$

Solution: Let the given statement be $P(n)$. Then,

$$P(n): 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2.$$

Putting $n = 1$ in the given statement, we get

$$\begin{aligned} \text{LHS} &= 1^3 = 1 \text{ and RHS} = \left(\frac{1 \times 2}{2} \right)^2 = 1^2 = 1. \\ \therefore \text{LHS} &= \text{RHS}. \end{aligned}$$

Thus, $P(1)$ is true.

Let $P(k)$ be true for some $k \in N$. Then,

$$P(k): 1^3 + 2^3 + 3^3 + \cdots + k^3 = \left\{ \frac{k(k+1)}{2} \right\}^2. \quad (1)$$

Now, $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$

$$= \{1^3 + 2^3 + 3^3 + \cdots + k^3\} + (k+1)^3$$

$$= \left\{ \frac{k(k+1)}{2} \right\}^2 + (k+1)^3$$

[using (i)]

$$\begin{aligned}
&= (k+1)^2 \left\{ \frac{k^2}{4} + (k+1) \right\} = (k+1)^2 \left\{ \frac{k^2 + 4k + 4}{4} \right\} \\
&= \frac{(k+1)^2 (k+2)^2}{4} = \left\{ \frac{(k+1)\{(k+1)+1\}}{2} \right\}^2 \\
\therefore P(k+1): 1^3 + 2^3 + 3^3 + \dots + (k+1)^3 &= \left[\frac{(k+1)\{(k+1)+1\}}{2} \right]^2.
\end{aligned}$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2 \text{ for all } n \in N.$$

Example 5: Using the principle of mathematical induction, prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

Solution: Let the given statement be $P(n)$. Then,

$$P(n): 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

When $n = 1$, LHS = $1 \cdot 2 = 2$ and RHS = $\frac{1}{3} \times 1 \times 2 \times (1+2) = 2$.

$\therefore$ LHS = RHS.

Thus, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + k(k+1) = \frac{1}{3}k(k+1)(k+2). \quad (1)$$

Now, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + k(k+1) + (k+1)(k+2)$

$$\begin{aligned}
&= \{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) + (k+1)(k+2)\} \\
&= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \quad [\text{using (1)}] \\
&= \frac{1}{3} \cdot [k(k+1)(k+2) + 3(k+1)(k+2)] = \frac{1}{3}(k+1)(k+2)(k+3). \\
\therefore P(k+1): 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (k+1)(k+2) \\
&= \frac{1}{3}(k+1)(k+2)(k+3).
\end{aligned}$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2) \text{ for all } n \in N.$$

Check Your Progress

1. Using mathematical induction, prove that $1 + 2 + 3 + \dots + n = n(n+1)/2$ for every positive integer n .

.....

.....

.....

2. Using mathematical induction, prove that $2^n \geq n + 1$ for every positive integer n .

.....

.....

.....

3. Using mathematical induction, prove that $3^n - 1$ is divisible by 2 for every positive integer n .

.....

.....

.....

Example 6: Using the principle of mathematical induction, prove that

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n - 1)(2n + 1) = \frac{1}{3}n(4n^2 + 6n - 1).$$

Solution: Let the given statement be $P(n)$. Then,

$$\begin{aligned} P(n): 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n - 1)(2n + 1) \\ = \frac{1}{3}n(4n^2 + 6n - 1). \end{aligned}$$

When $n = 1$, we have

$$\begin{aligned} \text{LHS} &= 1 \cdot 3 = 3 \text{ and } \text{RHS} = \frac{1}{3} \times 1 \times (4 \times 1^2 + 6 \times 1 - 1) = \frac{1}{3} \times 1 \times 9 = 3. \\ \therefore \text{LHS} &= \text{RHS}. \end{aligned}$$

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2k - 1)(2k + 1) = \frac{1}{3}k(4k^2 + 6k - 1). \quad (1)$$

Now, $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2k - 1)(2k + 1) + \{2(k + 1) - 1\}\{2(k + 1) + 1\}$

$$\begin{aligned} &= \{1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2k - 1)(2k + 1)\} + (2k + 1)(2k + 3) \\ &= \frac{1}{3}k(4k^2 + 6k - 1) + (2k + 1)(2k + 3) \end{aligned}$$

[using (1)]

$$\begin{aligned}
&= \frac{1}{3}[(4k^3 + 6k^2 - k) + 3(4k^2 + 8k + 3)] \\
&= \frac{1}{3}(4k^3 + 18k^2 + 23k + 9) \\
&= \frac{1}{3}(k+1)(4k^2 + 14k + 9) = \frac{1}{3}(k+1)[4(k+1)^2 + 6(k+1) - 1].
\end{aligned}$$

$$\begin{aligned}
\therefore P(k+1): 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + \{2(k+1) - 1\}\{2(k+1) + 1\} \\
= \frac{1}{3}(k+1)\{4(k+1)^2 + 6(k+1) - 1\}.
\end{aligned}$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\begin{aligned}
1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1) \\
= \frac{1}{3}n(4n^2 + 6n - 1) \text{ for all } n \in N.
\end{aligned}$$

Example 7: Using the principle of mathematical induction, prove that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

for all $n \in N$.

Solution: Let the given statement be $P(n)$. Then,

$$\begin{aligned}
P(n): 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) \\
= \frac{1}{4}n(n+1)(n+2)(n+3).
\end{aligned}$$

When $n = 1$, we have

$$\begin{aligned}
\text{LHS} &= 1 \cdot 2 \cdot 3 = 6 \text{ and RHS} = \frac{1}{4} \times 1 \times 2 \times 3 \times 4 = 6. \\
\therefore \text{LHS} &= \text{RHS}.
\end{aligned}$$

Thus, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$\begin{aligned}
 P(k): & 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) \\
 & = \frac{1}{4}k(k+1)(k+2)(k+3).
 \end{aligned} \tag{1}$$

Now, $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$$\begin{aligned}
 & = \{1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2)\} + (k+1)(k+2)(k+3) \\
 & = \frac{1}{4}k(k+1)(k+2)(k+3) + \frac{4(k+1)(k+2)(k+3)}{4} \quad [\text{using (i)}] \\
 & = \frac{1}{4}(k+1)(k+2)(k+3)(k+4).
 \end{aligned}$$

[using (1)]

$$\begin{aligned}
 \therefore P(k+1): & 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (k+1)(k+2)(k+3) \\
 & = \frac{1}{4}(k+1)(k+2)(k+3)(k+4).
 \end{aligned}$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

for all values of $n \in N$.

Example 8: Using the principle of mathematical induction, prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}.$$

Solution: Let the given statement be $P(n)$. Then,

$$P(n): \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}.$$

Putting $n = 1$ in the given statement, we get

$$\begin{aligned}
 \text{LHS} & = \frac{1}{1 \cdot 2} = \frac{1}{2} \text{ and RHS} = \frac{1}{(1+1)} = \frac{1}{2}. \\
 \therefore \text{LHS} & = \text{RHS}.
 \end{aligned}$$

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{(k+1)}. \quad (1)$$

$$\begin{aligned} \text{Now, } \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\ = \left\{ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k+1)} \right\} + \frac{1}{(k+1)(k+2)} \\ = \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} \end{aligned}$$

[using (1)]

$$= \frac{k(k+2) + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)}.$$

$$\therefore P(k+1): \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)}.$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \text{ for all } n \in N.$$

Example 9: Using the principle of mathematical induction, prove that

$$\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}.$$

Solution: Let the given statement be $P(n)$. Then,

$$P(n): \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}.$$

Putting $n = 1$ in the given statement, we get

$$\begin{aligned} \text{LHS} &= \frac{1}{3 \cdot 5} = \frac{1}{15} \text{ and } \text{RHS} = \frac{1}{3(2 \times 1 + 3)} = \frac{1}{15}. \\ \therefore \text{LHS} &= \text{RHS}. \end{aligned}$$

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)}. \quad (1)$$

$$\begin{aligned} \text{Now, } \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2k+1)(2k+3)} + \frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}} \\ &= \left\{ \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2k+1)(2k+3)} \right\} + \frac{1}{(2k+3)(2k+5)} \\ &= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)} \quad [\text{using (1)}] \\ &= \frac{k(2k+5) + 3}{3(2k+3)(2k+5)} = \frac{(2k^2 + 5k + 3)}{3(2k+3)(2k+5)} = \frac{(k+1)(2k+3)}{3(2k+3)(2k+5)} \\ &= \frac{(k+1)}{3(2k+5)} = \frac{(k+1)}{3\{2(k+1)+3\}}. \end{aligned}$$

$$\begin{aligned} \therefore P(k+1): \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}} \\ &= \frac{(k+1)}{3\{2(k+1)+3\}}. \end{aligned}$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true. $\therefore P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true. Hence, by the principle of mathematical induction, we have

$$\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

for all values of $n \in N$.

Example 10: Using the principle of mathematical induction, prove that

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)} \text{ for all } n \in \mathbb{N}.$$

Solution: Let the given statement be $P(n)$. Then,

$$P(n): \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS} = \frac{1}{1 \cdot 2 \cdot 3} = \frac{1}{6} \text{ and } \text{RHS} = \frac{1 \times (1+3)}{4 \times (1+1)(1+2)} = \frac{1 \times 4}{4 \times 2 \times 3} = \frac{1}{6}.$$

$$\therefore \text{LHS} = \text{RHS}.$$

Thus, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}. \quad (1)$$

$$\begin{aligned} \text{Now, } & \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \\ &= \left\{ \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{k(k+1)(k+2)} \right\} + \frac{1}{(k+1)(k+2)(k+3)} \\ &= \left\{ \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \right\} \quad [\text{using (1)}] \\ &= \frac{k(k+3)^2 + 4}{4(k+1)(k+2)(k+3)} = \frac{(k^3 + 6k^2 + 9k + 4)}{4(k+1)(k+2)(k+3)} \\ &= \frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}. \\ P(k+1): & \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{(k+1)(k+2)(k+3)} \\ \therefore &= \frac{(k+1)(k+4)}{4(k+2)(k+3)}. \end{aligned}$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true. $\therefore P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)}$$

$$= \frac{n(n+3)}{4(n+1)(n+2)}$$

for all values of $n \in \mathbb{N}$.

4.4 SUMMARY

So, in this unit we have learned about the Principle of Mathematical Induction. We also discussed various examples to explore the induction. This unit introduces the **Principle of Mathematical Induction (PMI)**, an important technique for proving mathematical statements involving positive integers.

- It explains the basic idea and logical foundation of **mathematical induction**.
- The **Base Step** is used to verify that the statement is true for the initial value.
- The **Inductive Step** involves assuming the statement is true for an arbitrary positive integer and proving that it is true for the next integer.
- The unit explains how these two steps work together to establish the truth of a statement for **all positive integers**.
- Various **examples and problems** are discussed to illustrate the application of mathematical induction.
- The unit also helps learners develop a **systematic and logical approach to mathematical proofs**.

- By the end of the unit, learners should be able to **understand, apply, and use the Principle of Mathematical Induction** to prove suitable mathematical results.

4.5 GLOSSARY

Mathematical Induction: A method for proving statements for all positive integers.

Proposition: A mathematical statement that can be proved true or false.

Base Case: The initial case verified in an induction proof.

Induction Hypothesis: The assumption that the statement holds for an arbitrary integer.

Induction Step: The step proving the statement for the next integer.

Successor: The integer immediately following a given integer.

Inductive Proof: A proof using a base case and an induction step.

Principle of Induction: A principle stating that base case plus induction step establishes a result for all relevant integers.

4.6 EXERCISES

By using principle of mathematical induction show that.

1. $1^4 + 2^4 + 3^4 + \dots + n^4 = n(n+1)(2n+1) \frac{(3n^2+3n-1)}{30}$.
2. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.
3. $1 + r + r^2 + \dots + r^n = \frac{1-r^{n+1}}{1-r}, \forall n \in \mathbb{N}, (r \neq 1)$.
4. $7^n - 4^n$ is a multiple of 3 for all $n \in \mathbb{N}$.
5. $5^{2n} - 1$ is a multiple of 8.
6. $9^n - 4^n$ is a multiple of 5.

4.7 SUGGESTED READINGS

1. David Burton, M. (2007). *Elementary Number Theory*. Seventh Edition.
2. Rosen, K. H. (2011). *Elementary Number Theory*. London: Pearson Education.
3. Hari Kishan. (2014). *Number Theory*. Krishna Prakashan. Meerut.
4. Hari Kishan. Shiv Raj Pundir. *Discrete Mathematics*. Pragati Prakashan, Meerut.



BLOCK-II: INTRODUCTION TO MATRIX THEORY

UNIT 5 MATRICES: TYPES AND BASIC PROPERTIES

STRUCTURE

- 5.1. Introduction
 - 5.2. Objectives
 - 5.3. Matrix: Definition and Types
 - 5.4. Matrix Addition
 - 5.5. Summary
 - 5.6. Glossary
 - 5.7. Exercises
 - 5.8. Suggested Readings
-

5.1 INTRODUCTION

Welcome to the unit on Matrices - Types and Basic Properties! In this unit, we will explore the fundamental concepts surrounding matrices, their various types, and the essential properties that govern their behavior. Matrices are foundational mathematical objects that find applications in diverse fields ranging from mathematics and physics to computer science, engineering, and economics.

Why Study Matrices? Matrices provide a structured way to represent and manipulate complex data and relationships. Understanding matrices is crucial for solving systems of linear equations, analyzing transformations, and processing data efficiently in numerous real-world applications.

5.2 OBJECTIVES

After going through this unit, students will be able to

- Define matrix and understand related terms
- Differentiate between various types of matrices
- Do addition of matrices

5.3 MATRIX: DEFINITION AND TYPES

Matrix: A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns) is called a matrix of order m by n , written as an $m \times n$ matrix.

Such an array is enclosed by $[]$ or $()$.

Each of the mn numbers constituting the matrix is called an element or an entry of the matrix.

Usually, we denote a matrix by a capital letter.

The plural of matrix is matrices.

Example

(i) $A = \begin{bmatrix} 3 & 5 & -4 \\ 0 & 1 & 9 \end{bmatrix}$ is a matrix, having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements.

(ii) $B = \begin{bmatrix} 9 & 4 & \sqrt{2} & -1 \\ 1 & 8 & -3 & 2 \\ 6 & 0 & 5 & 7 \end{bmatrix}$ is a matrix, having 3 rows and 4 columns. Its order is 3×4 and it has 12 elements.

How to Describe a Matrix?

In order to locate the position of a particular element of a matrix, we have to specify the number of the row and that of the column in which the element occurs.

An element occurring in the i th row and j th column of a matrix A will be called the (i, j) th element of A , to be denoted by a_{ij} .

In general, an $m \times n$ matrix A may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}.$$

Example 1: Consider the matrix $A = \begin{bmatrix} 3 & -2 & 5 \\ 6 & 9 & 1 \end{bmatrix}$.

Clearly, the element in the 1st row and 2nd column is -2. So, we write $a_{12} = -2$.

Similarly, $a_{11} = 3; a_{12} = -2; a_{13} = 5; a_{21} = 6; a_{22} = 9$ and $a_{23} = 1$.

Example 2: Construct a 3×2 matrix whose elements are given by $a_{ij} = (i + 2j)$.

Solution: A 3×2 matrix has 3 rows and 2 columns.

In general, a 3×2 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3 \times 2}$$

Thus $a_{ij} = (i + 2j)$ for $i = 1, 2, 3$ and $j = 1, 2$

$$\therefore a_{11} = (1 + 2 \times 1) = 3; a_{12} = (1 + 2 \times 2) = 5;$$

$$a_{21} = (2 + 2 \times 1) = 4; a_{22} = (2 + 2 \times 2) = 6;$$

$$a_{31} = (3 + 2 \times 1) = 5; a_{32} = (3 + 2 \times 2) = 7.$$

$$\text{Hence, } A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{bmatrix}_{3 \times 2}.$$

Example 3: Construct a 2×3 matrix whose elements are given by $a_{ij} = \frac{1}{2}|5i - 3j|$.

Solution: A 2×3 matrix has 2 rows and 3 columns.

In general, a 2×3 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3}$$

Thus, $a_{ij} = \frac{1}{2}|5i - 3j|$ where $i = 1, 2$ and $j = 1, 2, 3$.

$$\begin{aligned}
 \therefore a_{11} &= \frac{1}{2} |5 \times 1 - 3 \times 1| = \frac{1}{2} \cdot |2| = \frac{1}{2} \times 2 = 1; \\
 a_{12} &= \frac{1}{2} |5 \times 1 - 3 \times 2| = \frac{1}{2} \cdot |5 - 6| = \frac{1}{2} \cdot |-1| = \frac{1}{2} \times 1 = \frac{1}{2}; \\
 a_{13} &= \frac{1}{2} |5 \times 1 - 3 \times 3| = \frac{1}{2} \cdot |5 - 9| = \frac{1}{2} \cdot |-4| = \frac{1}{2} \times 4 = 2; \\
 a_{21} &= \frac{1}{2} |5 \times 2 - 3 \times 1| = \frac{1}{2} \cdot |10 - 3| = \frac{1}{2} \cdot |7| = \frac{1}{2} \times 7 = \frac{7}{2}; \\
 a_{22} &= \frac{1}{2} |5 \times 2 - 3 \times 2| = \frac{1}{2} \cdot |10 - 6| = \frac{1}{2} \cdot |4| = \frac{1}{2} \times 4 = 2; \\
 a_{23} &= \frac{1}{2} |5 \times 2 - 3 \times 3| = \frac{1}{2} \cdot |10 - 9| = \frac{1}{2} \cdot |1| = \frac{1}{2} \times 1 = \frac{1}{2}.
 \end{aligned}$$

Hence, $A = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 7 & 2 & \frac{1}{2} \end{bmatrix}$.

Example 4: If a matrix has 12 elements, what are the possible orders it can have?

Solution: We know that a matrix of order $m \times n$ has mn elements.

Hence, all possible orders of a matrix having 12 elements are

$(12 \times 1), (1 \times 12), (6 \times 2), (2 \times 6), (4 \times 3)$ and (3×4) .

Types of Matrices

Row Matrix: A matrix having only one row is known as a row matrix or a row vector.

Examples

(i) $A = [5 \quad 18]$ is a row matrix of order 1×2 .

Column Matrix: A matrix having only one column is known as a column matrix or a column vector.

Examples

(i) $A = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a column matrix of order 3×1 .

(ii) $B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ is a column matrix of order 2×1 .

Zero or Null Matrix: A matrix each of whose elements is zero is called a zero matrix or a null matrix.

Examples The matrices $[0]$, $[0 \ 0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are null matrices of order (1×1) , (1×2) , (2×2) and (2×3) respectively.

Square Matrix: A matrix having the same number of rows and columns is called a square matrix.

A matrix of order $(n \times n)$ is called a square matrix of order n or an n -rowed square matrix.

A matrix of order $m \times n$, where $m \neq n$, is called a rectangular matrix.

Examples

(i) The matrix $\begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$ is a 2-rowed square matrix.

(ii) The matrix $\begin{bmatrix} 5 & 3 & 6 \\ 7 & \sqrt{2} & -4 \\ -9 & \frac{1}{3} & 0 \end{bmatrix}$ is a 3-rowed square matrix.

Diagonal Elements of a Matrix: Let $A = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix.

Then, the elements a_{ij} for which $i = j$, are called the diagonal elements of A .

Thus, the diagonal elements of $A = [a_{ij}]_{m \times n}$ are $a_{11}, a_{22}, a_{33}, a_{44}$, etc.

The line along which the diagonal elements lie is called the diagonal of the matrix.

Example Let $A = \begin{bmatrix} 3 & 2 & -1 \\ \sqrt{5} & \frac{5}{8} & 7 \\ 6 & -4 & \sqrt{2} \end{bmatrix}$

Then, the diagonal elements of A are:

$$a_{11} = 3, a_{22} = \frac{5}{8}, a_{33} = \sqrt{2}.$$

Diagonal Matrix: A square matrix in which every nondiagonal element is zero is called a diagonal matrix.

If $A = [a_{ij}]_{n \times n}$ be a diagonal matrix then $a_{ij} = 0$ when $i \neq j$ and we write it as

$$A = \text{diag} [a_{11}, a_{22}, a_{33}, \dots, a_{nn}].$$

Example Let $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Then, A is a diagonal matrix.

We may write it as, $A = \text{diag} [6, 4, -2]$.

Scalar Matrix: A square matrix in which every nondiagonal element is zero and all diagonal elements are equal is known as a scalar matrix.

Examples

(i) $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ is a scalar matrix of order 2 .

(ii) $B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a scalar matrix of order 3 .

Unit Matrix: A square matrix in which every nondiagonal element is 0 and every diagonal element is 1 is called a unit matrix or an identity matrix.

Thus, a square matrix $[a_{ij}]_{n \times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 0 & \text{when } i \neq j, \\ 1 & \text{when } i = j. \end{cases}$$

A unit matrix of order n will be denoted by I_n or simply by I .

Examples

(i) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

(ii) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3 .

Comparable Matrices: Two matrices A and B are said to be comparable if they are of the same order, i.e., they have the same number of rows and the same number of columns.

Example $A = \begin{bmatrix} 2 & -5 & 1 \\ 0 & 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 & 0 \\ 1 & 4 & -9 \end{bmatrix}$ are comparable matrices, each being of order (2×3) .

Equal Matrices: Two matrices A and B are said to be equal, written as $A = B$, if they are of the same order and their corresponding elements are equal.

Example 1: Find x, y, z when $\begin{bmatrix} 5 & 3 \\ x & 7 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 7 \end{bmatrix}$.

Solution: Since the corresponding elements of equal matrices are equal, we have

$$\begin{bmatrix} 5 & 3 \\ x & 7 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 7 \end{bmatrix} \Leftrightarrow x = 1, y = 5 \text{ and } z = 3.$$

Example 2: Find x, y, z, w when $\begin{bmatrix} x - y & 2x + z \\ 2x - y & 3z + w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$.

Solution: We know that in equal matrices, the corresponding elements are equal.

$$\begin{aligned} \therefore \begin{bmatrix} x - y & 2x + z \\ 2x - y & 3z + w \end{bmatrix} &= \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix} \\ \Leftrightarrow x - y &= -1, 2x - y = 0, 2x + z = 5 \text{ and } 3z + w = 13. \end{aligned}$$

Solving the first two equations, we get $x = 1$ and $y = 2$.

Putting $x = 1$ in $2x + z = 5$, we get $z = 3$.

Putting $z = 3$ in $3z + w = 13$, we get $w = 4$.

$\therefore x = 1, y = 2, z = 3$ and $w = 4$.

Example 3: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Why?

Solution: Since the given null matrices are not comparable, they are not equal.

Check Your Progress

1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}$, find $A + B$ and $2A - B$.

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.....

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2. Find the order and transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

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.....

.....

3. Determine whether $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is a diagonal and scalar matrix. State the result from the entries.

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5.4 ADDITION OF MATRICES

Let A and B be two comparable matrices, each of order $(m \times n)$. Then, their sum $(A + B)$ is a matrix of order $(m \times n)$, obtained by adding the corresponding elements of A and B .

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then

$$A + B = [a_{ij} + b_{ij}]_{m \times n}$$

Remark: For two matrices A and B , the sum $(A + B)$ exists only when A and B are comparable.

Example 1: If $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ then A and B are matrices of order 2×2 and 2×3 respectively. So, A and B are not comparable. Hence, $A + B$ is not defined.

Example 2: Let $A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 2 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$. Clearly, each one of A and B is a 2×3 matrix. So, A and B are comparable matrices. $\therefore A + B$ is defined.

$$\begin{aligned} \text{Now, } A + B &= \begin{bmatrix} 5 + 4 & 0 + (-3) & -2 + (-6) \\ 3 + (-1) & 2 + 0 & -7 + 4 \end{bmatrix} \\ &= \begin{bmatrix} 9 & -3 & -8 \\ 2 & 2 & -3 \end{bmatrix}. \end{aligned}$$

Some Results on Addition of Matrices

Theorem 1: Matrix addition is commutative, i.e., $A + B = B + A$ for all comparable matrices A and B .

Proof: Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then,

$$\begin{aligned} A + B &= [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} \\ &= [a_{ij} + b_{ij}]_{m \times n} \text{ [by the definition of addition of matrices]} \\ &= [b_{ij} + a_{ij}]_{m \times n} \text{ [} \because \text{ addition of numbers is commutative]} \\ &= [b_{ij}]_{m \times n} + [a_{ij}]_{m \times n} = B + A. \end{aligned}$$

Hence, $A + B = B + A$.

Theorem 2: Matrix addition is commutative,

i.e., $(A + B) + C = A + (B + C)$ for all comparable matrices A, B, C .

Proof: Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ and $C = [c_{ij}]_{m \times n}$. Then,

$$\begin{aligned}
(A + B) + C &= ([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}) + [c_{ij}]_{m \times n} \\
&= [a_{ij} + b_{ij}]_{m \times n} + [c_{ij}]_{m \times n} \\
&= [(a_{ij} + b_{ij}) + c_{ij}]_{m \times n} \\
&= [a_{ij} + (b_{ij} + c_{ij})]_{m \times n} \\
&\quad [\because \text{addition of numbers is associative}]
\end{aligned}$$

$$\begin{aligned}
&= [a_{ij}]_{m \times n} + [b_{ij} + c_{ij}]_{m \times n} \\
&= [a_{ij}]_{m \times n} + ([b_{ij}] + [c_{ij}]_{m \times n}) = A + (B + C).
\end{aligned}$$

Hence, $(A + B) + C = A + (B + C)$.

Theorem 3: If A is an $m \times n$ matrix and O is an $m \times n$ null matrix then

$$A + O = O + A = A.$$

Proof: Let $A = [a_{ij}]_{m \times n}$ and $O = [b_{ij}]_{m \times n}$,

where $b_{ij} = 0$ for all suffixes i and j .

Then,

$$\begin{aligned}
A + O &= [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n} \\
&= [a_{ij} + 0]_{m \times n} [\because b_{ij} = 0] \\
&= [a_{ij}]_{m \times n} = A. \\
A + O &= A.
\end{aligned}$$

$\therefore A + O = A$.

Similarly, $O + A = A$.

Hence, $A + O = O + A = A$.

Remark: The null matrix O of order $m \times n$ is the additive identity in the set of all $m \times n$ matrices.

Example 3: Let $A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then verify that

$$A + O = O + A = A.$$

Solution: Clearly, each one of A and O is a matrix of order (2×3) .

So, $(A + O)$ and $(O + A)$ are both defined.

$$\begin{aligned} \text{Now, } A + O &= \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3+0 & 5+0 & 4+0 \\ 1+0 & 2+0 & -3+0 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} = A. \end{aligned}$$

And,

$$\begin{aligned} O + A &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 0+3 & 0+5 & 0+4 \\ 0+1 & 0+2 & 0+(-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} = A. \end{aligned}$$

Hence, $A + O = O + A = A$.

Negative of a Matrix: Let $A = [a_{ij}]_{m \times n}$. Then, the negative of A is the matrix $(-A) = [-a_{ij}]_{m \times n}$, obtained by replacing each element of A with its corresponding additive inverse. $(-A)$ is called the additive inverse of A .

Example 4: If $A = \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix}$, find $(-A)$ and verify that

$$A + (-A) = (-A) + A = O.$$

Solution: Clearly, we have:

$$\begin{aligned} (-A) &= \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix} \\ A + (-A) &= \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix} + \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 3+(-3) & -2+2 & 0+0 \\ -5+5 & 7+(-7) & \sqrt{2}+(-\sqrt{2}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O. \end{aligned}$$

$$\begin{aligned} \text{And, } (-A) + A &= \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix} + \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} -3+3 & 2+(-2) & 0+0 \\ 5+(-5) & -7+7 & -\sqrt{2}+\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Hence, $A + (-A) = (-A) + A = O$.

Solved Examples

Example 1: Let $A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix}$. Verify that $A + B = B + A$.

Solution: Here, A is a 2×3 matrix and B is a 2×3 matrix. So, A and B are comparable.

Therefore, $(A + B)$ and $(B + A)$ both exist and each is a 2×3 matrix. Now,

$$\begin{aligned} A + B &= \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2+4 & 3+(-2) & 5+3 \\ -1+2 & 0+6 & 4+(-1) \end{bmatrix} = \begin{bmatrix} 6 & 1 & 8 \\ 1 & 6 & 3 \end{bmatrix}. \end{aligned}$$

And,

$$\begin{aligned} B + A &= \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4+2 & -2+3 & 3+5 \\ 2+(-1) & 6+0 & (-1)+4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 8 \\ 1 & 6 & 3 \end{bmatrix}. \end{aligned}$$

Hence, $A + B = B + A$.

Example 2: Let $A = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$.

Verify that $(A + B) + C = A + (B + C)$.

Solution: Clearly, each one of the matrices A, B, C is a (3×2) matrix. So, $(A + B) + C$ and $A + (B + C)$ are both defined and each one is a 3×2 matrix.

$$\text{Now, } (A + B) = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} 1+3 & -2+1 \\ 5+0 & 4+2 \\ 3+(-3) & 0+5 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 5 & 6 \\ 0 & 5 \end{bmatrix} \\
 \therefore (A+B)+C &= \begin{bmatrix} 4 & -1 \\ 5 & 6 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix} \\
 &= \begin{bmatrix} 4+4 & -1+3 \\ 5+(-2) & 6+2 \\ 0+1 & 5+6 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 8 \\ 1 & 11 \end{bmatrix} \\
 \text{Also, } (B+C) &= \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix} \\
 &= \begin{bmatrix} 3+4 & 1+3 \\ 0+(-2) & 2+2 \\ -3+1 & 5+6 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ -2 & 4 \\ -2 & 11 \end{bmatrix} \\
 \therefore A+(B+C) &= \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ -2 & 4 \\ -2 & 11 \end{bmatrix} \\
 &= \begin{bmatrix} 1+7 & -2+4 \\ 5+(-2) & 4+4 \\ 3+(-2) & 0+11 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 8 \\ 1 & 11 \end{bmatrix}
 \end{aligned}$$

Hence, $(A+B)+C = A+(B+C)$.

Example 3: Find the additive inverse of the matrix $A = \begin{bmatrix} 2 & -5 & 0 \\ 4 & 3 & -1 \end{bmatrix}$.

Solution: Clearly, the additive inverse of the given matrix A is the matrix $-A$, given by

$$-A = \begin{bmatrix} -2 & -(-5) & 0 \\ -4 & -3 & -(-1) \end{bmatrix} = \begin{bmatrix} -2 & 5 & 0 \\ -4 & -3 & 1 \end{bmatrix}.$$

Subtraction of Matrices: If A and B are two comparable matrices then we define

$$(A - B) = A + (-B).$$

Example 4: If $A = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 7 & -9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 8 & -4 \end{bmatrix}$, find $(A - B)$.

Solution: We have, $(-B) = \begin{bmatrix} -1 & -2 & 3 \\ -4 & -8 & 4 \end{bmatrix}$.

$$\begin{aligned}
 \therefore (A - B) &= A + (-B) \\
 &= \begin{bmatrix} 2 & -3 & 1 \\ 0 & 7 & -9 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 3 \\ -4 & -8 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 2 + (-1) & -3 + (-2) & 1 + 3 \\ 0 + (-4) & 7 + (-8) & -9 + 4 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 4 \\ -4 & -1 & -5 \end{bmatrix}.
 \end{aligned}$$

Hence, $(A - B) = \begin{bmatrix} 1 & -5 & 4 \\ -4 & -1 & -5 \end{bmatrix}$.

Example 5: If $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$, then find the value of $(a - 2b)$.

Solution: Comparing the corresponding elements of given equal matrices, we have $a + 4 = 2a + 2$, $3b = b + 2$ and $a - 8b = -6$.
From these equations, we get $a = 2$ and $b = 1$.

$$\therefore (a - 2b) = (2 - 2 \times 1) = (2 - 2) = 0.$$

Example 6: If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix A .

Solution: Clearly, we have

$$\begin{aligned}
 A &= \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix} \\
 &= \begin{bmatrix} 9-1 & -1-2 & 4+1 \\ -2-0 & 1-4 & 3-9 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}.
 \end{aligned}$$

Scalar Multiplication: Let A be a given matrix and k be a number. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k , to be denoted by kA .

If A is an $(m \times n)$ matrix then kA is also an $(m \times n)$ matrix.

If $A = [a_{ij}]_{m \times n}$ then $kA = [k \cdot a_{ij}]_{m \times n}$.

Example 7 If $A = \begin{bmatrix} 5 & 4 & -2 \\ 6 & -1 & 7 \end{bmatrix}$, find (i) $3A$

(ii) $\frac{1}{2}A$

(iii) $-2A$.

Solution: We have:

(i) $3A = \begin{bmatrix} 3 \cdot 5 & 3 \cdot 4 & 3 \cdot (-2) \\ 3 \cdot 6 & 3 \cdot (-1) & 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 & -6 \\ 18 & -3 & 21 \end{bmatrix}.$

$$\begin{aligned}
 \text{(ii)} \quad \frac{1}{2}A &= \begin{bmatrix} \frac{1}{2} \cdot 5 & \frac{1}{2} \cdot 4 & \frac{1}{2} \cdot (-2) \\ \frac{1}{2} \cdot 6 & \frac{1}{2} \cdot (-1) & \frac{1}{2} \cdot 7 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 2 & -1 \\ 3 & -\frac{1}{2} & \frac{7}{2} \end{bmatrix} \\
 \text{(iii)} \quad -2A &= (-2) \cdot A = \begin{bmatrix} (-2) \cdot 5 & (-2) \cdot 4 & (-2) \cdot (-2) \\ (-2) \cdot 6 & (-2) \cdot (-1) & (-2) \cdot 7 \end{bmatrix} \\
 &= \begin{bmatrix} -10 & -8 & 4 \\ -12 & 2 & -14 \end{bmatrix}.
 \end{aligned}$$

Properties of Scalar Multiplication

Theorem 1: If A and B are two matrices of the same order and k is a scalar then prove that $k(A + B) = kA + kB$.

Proof: Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then,

$$\begin{aligned}
 k(A + B) &= k \cdot ([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}) \\
 &= k \cdot [a_{ij} + b_{ij}]_{m \times n} \quad [\text{by definition of addition of matrices}] \\
 &= [k(a_{ij} + b_{ij})]_{m \times n} \quad [\text{by definition of scalar multiplication}] \\
 &= [k \cdot a_{ij} + k \cdot b_{ij}]_{m \times n} \quad [\text{by distributive law}] \\
 &= [k \cdot a_{ij}]_{m \times n} + [k \cdot b_{ij}]_{m \times n} \\
 &= kA + kB.
 \end{aligned}$$

Hence, $k(A + B) = kA + kB$.

Theorem 2: If A is any matrix and k_1, k_2 are any scalars then prove that

$$(i) \quad (k_1 + k_2)A = k_1A + k_2A$$

$$(ii) \quad k_1(k_2A) = (k_1k_2)A.$$

Proof: Let $A = [a_{ij}]_{m \times n}$. Then,

(i)

$$\begin{aligned}
 (k_1 + k_2)A &= (k_1 + k_2) \cdot [a_{ij}]_{m \times n} \\
 &= [(k_1 + k_2) \cdot a_{ij}]_{m \times n} \quad [\text{by definition of scalar multiplication}] \\
 &= [k_1 \cdot a_{ij} + k_2 \cdot a_{ij}]_{m \times n} \quad [\text{by distributive law}] \\
 &= [k_1 \cdot a_{ij}]_{m \times n} + [k_2 \cdot a_{ij}]_{m \times n}
 \end{aligned}$$

[by definition of addition of matrices]

$$= k_1 A + k_2 A.$$

Hence, $(k_1 + k_2)A = k_1 A + k_2 A$.

(ii)

$$\begin{aligned} k_1(k_2 A) &= k_1[k_2 \cdot a_{ij}]_{m \times n} = [k_1(k_2 \cdot a_{ij})]_{m \times n} \\ &= [(k_1 k_2) \cdot a_{ij}]_{m \times n} \end{aligned}$$

[by associativity of multiplication in numbers]

$$= (k_1 k_2) \cdot [a_{ij}]_{m \times n} = (k_1 k_2) A.$$

Hence, $k_1(k_2 A) = (k_1 k_2)A$.

Solved Examples

Example 1: If $A = \begin{bmatrix} 3 & 5 \\ 7 & -9 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -4 \\ 2 & 3 \end{bmatrix}$, find $(4A - 3B)$.

Solution: $(4A - 3B) = 4A + (-3B)$.

$$\text{Now, } 4A = \begin{bmatrix} 4 \cdot 3 & 4 \cdot 5 \\ 4 \cdot 7 & 4 \cdot (-9) \end{bmatrix} = \begin{bmatrix} 12 & 20 \\ 28 & -36 \end{bmatrix}.$$

$$\text{And, } -3B = (-3) \cdot B = \begin{bmatrix} (-3) \cdot 6 & (-3) \cdot (-4) \\ (-3) \cdot 2 & (-3) \cdot 3 \end{bmatrix} = \begin{bmatrix} -18 & 12 \\ -6 & -9 \end{bmatrix}.$$

$$\begin{aligned} \therefore 4A - 3B &= 4A + (-3B) \\ &= \begin{bmatrix} 12 & 20 \\ 28 & -36 \end{bmatrix} + \begin{bmatrix} -18 & 12 \\ -6 & -9 \end{bmatrix} \\ &= \begin{bmatrix} 12 + (-18) & 20 + 12 \\ 28 + (-6) & -36 + (-9) \end{bmatrix} = \begin{bmatrix} -6 & 32 \\ 22 & -45 \end{bmatrix}. \end{aligned}$$

$$\text{Hence, } (4A - 3B) = \begin{bmatrix} -6 & 32 \\ 22 & -45 \end{bmatrix}.$$

Example 2: Let $A = \text{diag}[3, -5, 7]$ and $B = \text{diag}[-1, 2, 4]$.

Find (i) $(A + B)$ (ii) $(A - B)$ (iii) $-5A$ (iv) $(2A + 3B)$.

Solution: We have

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$\therefore \text{(i) } A + B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 11 \end{bmatrix}.$$

$$\text{(ii) } (A - B) = A + (-B)$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\text{(iii) } -5A = (-5) \cdot A = (-5) \cdot \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -35 \end{bmatrix}.$$

$$\text{(iv) } 2A + 3B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 26 \end{bmatrix}.$$

Example 3: Simplify $\cos \theta \cdot \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} + \sin \theta \cdot \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}.$

Solution: We have

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 4: If $2 \begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, find $(x - y)$.

Solution: We have

$$2 \begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6+1 & 8+y \\ 10+0 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow 2x + 1 = 5 \text{ and } 8 + y = 0$$

$$\Rightarrow x = 2 \text{ and } y = -8.$$

$$\Rightarrow (x - y) = 2 - (-8) = 10.$$

Example 5: Find a matrix X , if $X + \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$.

Then, the given matrix equation is $X + A = B$.

$$\begin{aligned}\therefore X + A = B &\Rightarrow X = B + (-A) \\ &= \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix} + \begin{bmatrix} -4 & -6 \\ 3 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 3 + (-4) & -6 + (-6) \\ 5 + 3 & -8 + (-7) \end{bmatrix} = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}.\end{aligned}$$

Hence, $X = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$.

Example 6: Find a matrix X such that $2A + B + X = O$, where $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ and

$$B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}.$$

Solution: We have

$$\begin{aligned}2A + B + X &= O \Rightarrow X = -(2A + B). \\ \text{Now, } (2A + B) &= 2 \cdot \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -2 + 3 & 4 + (-2) \\ 6 + 1 & 8 + 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 13 \end{bmatrix}. \\ \therefore X = -(2A + B) &= \begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}.\end{aligned}$$

Example 7: Find matrices X and Y , if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution: Adding the given matrices, we get

$$\begin{aligned}
 (X + Y) + (X - Y) &= \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix} \\
 \Rightarrow 2X &= \begin{bmatrix} 5+3 & 2+6 \\ 0+0 & 9+(-1) \end{bmatrix} \\
 \Rightarrow 2X &= \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow X = \frac{1}{2} \cdot \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}.
 \end{aligned}$$

On subtracting the given matrices, we get

$$\begin{aligned}
 (X + Y) - (X - Y) &= \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix} \\
 \Rightarrow 2Y &= \begin{bmatrix} 5-3 & 2-6 \\ 0-0 & 9-(-1) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} \\
 \Rightarrow Y &= \frac{1}{2} \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}.
 \end{aligned}$$

Hence, $X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$.

5.5 SUMMARY

So, in this first unit we have learned about:

- Matrix and their types
- When and how matrices can be added
- Matrix equation with addition and subtraction

5.6 GLOSSARY

Matrix: A rectangular arrangement of numbers or elements in rows and columns.

Order of Matrix: The number of rows and columns of a matrix.

Row Matrix: A matrix having only one row.

Column Matrix: A matrix having only one column.

Square Matrix: A matrix having the same number of rows and columns.

Zero Matrix: A matrix whose every element is zero.

Diagonal Matrix: A square matrix whose off-diagonal elements are zero.

Identity Matrix: A square matrix with ones on the main diagonal and zeros elsewhere.

5.7 EXERCISES

- 1 If $A = \begin{bmatrix} 4 & -2 & 6 & 1 \\ 7 & 5 & 8 & -3 \\ 9 & 8 & 4 & 3 \end{bmatrix}$ then write
 - (i) the number of rows in A ,
 - (ii) the number of columns in A ,
 - (iii) the order of the matrix A ,
 - (iv) the number of all entries in A ,
 - (v) the elements $a_{23}, a_{31}, a_{14}, a_{33}, a_{22}$ of A .
- 2 Write the order of each of the following matrices:
 - (i) $A = \begin{bmatrix} 1 & 5 & 4 & -2 \\ 0 & 9 & -1 & 5 \end{bmatrix}$
 - (ii) $B = \begin{bmatrix} 6 & -6 \\ \frac{1}{2} & \frac{3}{4} \\ 9 & -1 \end{bmatrix}$
 - (iii) $C = [7 \quad -12 \quad 5 \quad 0]$
 - (iv) $D = [8 \quad -2]$
 - (v) $E = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$
 - (vi) $F = [8]$
- 3 If a matrix has 12 elements, what are the possible orders it can have?
- 4 Find all possible orders of matrices having 7 elements.
- 5 Construct a 3×2 matrix whose elements are given by $a_{ij} = (2i + j)$.

- 6 Construct a 4×3 matrix whose elements are given by $a_{ij} = \frac{i+j}{j}$.
- 7 Construct a 2×2 matrix whose elements are $a_{ij} = \frac{(i+3j)^2}{2}$.
- 8 Construct a 2×3 matrix whose elements are $a_{ij} = \frac{(i-j)^2}{2}$.
- 9 Construct a 3×4 matrix whose elements are given by $a_{ij} = \frac{1}{2}| -3i + j|$.
- 10 If $A = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 2 & -2 \\ 4 & -3 & 1 \end{bmatrix}$, verify that $(A + B) = (B + A)$.
- 11 If $A = \begin{bmatrix} 3 & 5 \\ 4 & 0 \\ 6 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -3 \\ 3 & 2 \\ -2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 2 \\ 3 & -4 \\ 1 & 6 \end{bmatrix}$, verify that $(A + B) + C = A + (B + C)$.
- 12 If $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 & 4 \\ 5 & -3 & 2 \end{bmatrix}$, find $(2A - 3B)$.
- 13 Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 5 \\ 3 & 4 \end{bmatrix}$. Find:
- $A + 2B$
 - $B - 4C$
 - $A - 2B + 3C$
- 14 Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 5 & -1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 & -1 \\ 0 & -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -5 & 1 \\ -4 & 0 & 6 \end{bmatrix}$. Compute $5A - 3B + 4C$.
- 15 If $5A = \begin{bmatrix} 5 & 10 & -15 \\ 25 & 20 & 10 \\ 10 & 0 & -5 \end{bmatrix}$, find
- 16 Find matrices A and B , if

$$A + B = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 4 & -6 \\ 7 & 3 & 8 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} -5 & -4 & 8 \\ 13 & 2 & 0 \\ -1 & 7 & 4 \end{bmatrix}.$$

17 Find matrix X , if $\begin{bmatrix} 3 & 5 & -9 \\ -1 & 4 & -7 \end{bmatrix} + X = \begin{bmatrix} 6 & 2 & 3 \\ 4 & 8 & 6 \end{bmatrix}$.

18 If $A = \begin{bmatrix} -2 & 3 \\ 4 & 5 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ -7 & 3 \\ 6 & 4 \end{bmatrix}$, find a matrix C such that
 $A + B - C = O$.

19 Find the matrix X such that $2A - B + X = O$, where $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$
and $B = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$.

20 If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, find a matrix C such
that $(A + B + C)$ is a zero matrix.

21 If $A = \text{diag } [2, -5, 9]$, $B = \text{diag } [-3, 7, 14]$ and $C = \text{diag } [4, -6, 3]$,
find diagonal of
(i) $A - 2B$
(ii) $B + C - 2A$
(iii) $2A + B - 4C$.

22 Find the values of x and y , when

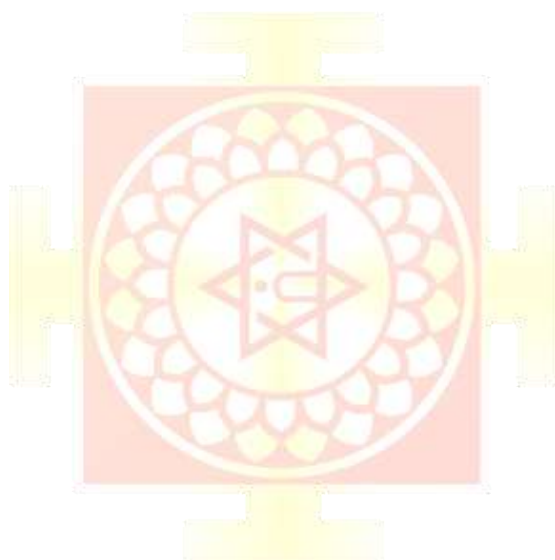
(i) $\begin{bmatrix} x + y \\ x - y \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 2x + 6 & 7 \\ 0 & 3y - 7 \end{bmatrix} = \begin{bmatrix} x - 3 & 7 \\ 0 & 5 \end{bmatrix}$

(iii) $2 \begin{bmatrix} x & 5 \\ 7 & y - 3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

5.8 SUGGESTED READINGS

1. Narayan Shanti and Mittal P.K., A Textbook of Matrices, S. Chand, 2010.
2. Hari Kishan, A Textbook of Matrices, Atlantic, 2023.



UNIT 6 MATRIX MULTIPLICATION AND DETERMINANT

STRUCTURE

- 6.1. Introduction
- 6.2. Objectives
- 6.3. Matrix Multiplication
- 6.4. Determinant
- 6.5. Summary
- 6.6. Glossary
- 6.7. Exercises
- 6.8. Suggested Readings

6.1 INTRODUCTION

Welcome to the world of matrices, where numbers are organized into rows and columns to solve complex problems and model real-world situations. In this unit, we will embark on a journey to explore two fundamental concepts in linear algebra: matrix multiplication and determinants.

Matrices serve as powerful tools in various fields, from mathematics and computer science to engineering and economics. Understanding how to manipulate matrices through multiplication opens doors to solving systems of equations, transforming geometric shapes, and analyzing data.

Additionally, determinants provide essential information about the properties and behavior of matrices. By learning to compute determinants, we gain insights into the invertibility of matrices and their geometric significance.

6.2 OBJECTIVES

The objective of this unit is to provide students with a fundamental understanding of matrix multiplication and determinants. By the end of this unit, students should be able:

- to perform matrix multiplication, compute determinants of matrices
- to understand their significance in various mathematical and real-world applications.

6.3 MATRIX MULTIPLICATION

For two given matrices A and B , we say that the product AB exists only when the number of rows in A equals the number of columns in B .

When AB exists, we say that A is conformable to B for multiplication.

$$\begin{aligned}\text{and } c_{22} &= (2 \text{ nd row of } A) \times (2 \text{ nd column of } B) \\ &= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}.\end{aligned}$$
$$\begin{aligned}\therefore AB &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}.\end{aligned}$$

Again, B is a 3×2 matrix and A is a 2×3 matrix. So, BA exists and it is a 3×3 matrix.

Proceeding as above, we get

$$\begin{aligned}BA &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\ &= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \end{bmatrix}.\end{aligned}$$

Example 1: If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$, find AB . Does BA exist?

Solution: Here A is a 3×2 matrix and B is a 2×2 matrix.

Clearly, the number of columns in A equals the number of rows in B .

$\therefore AB$ exists and it is a 3×2 matrix.

$$\begin{aligned} \text{Now, } AB &= \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot (-1) + (-1) \cdot 2 & 2 \cdot 3 + (-1) \cdot 1 \\ 3 \cdot (-1) + 4 \cdot 2 & 3 \cdot 3 + 4 \cdot 1 \\ 1 \cdot (-1) + 5 \cdot 2 & 1 \cdot 3 + 5 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 5 \\ 5 & 13 \\ 9 & 8 \end{bmatrix}. \end{aligned}$$

Further, B is a 2×2 matrix and A is a 3×2 matrix. So, the number of columns in B is not equal to the number of rows in A .

So, BA does not exist.

Example 2: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix}$. Find AB and BA ,

and show that $AB \neq BA$.

Solution: Here A is a 2×3 matrix and B is a 3×2 matrix.

So, AB exists and it is a 2×2 matrix.

Now,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 + (-2) \cdot 4 + 3 \cdot (-2) & 1 \cdot 3 + (-2) \cdot 5 + 3 \cdot 1 \\ (-4) \cdot 2 + 2 \cdot 4 + 5 \cdot (-2) & (-4) \cdot 3 + 2 \cdot 5 + 5 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} -12 & -4 \\ -10 & 3 \end{bmatrix}. \end{aligned}$$

Again, B is a 3×2 matrix and A is a 2×3 matrix.

So, BA exists and it is a 3×3 matrix.

Now,

$$\begin{aligned} BA &= \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 1 + 3 \cdot (-4) & 2 \cdot (-2) + 3 \cdot 2 & 2 \cdot 3 + 3 \cdot 5 \\ 4 \cdot 1 + 5 \cdot (-4) & 4 \cdot (-2) + 5 \cdot 2 & 4 \cdot 3 + 5 \cdot 5 \\ (-2) \cdot 1 + 1 \cdot (-4) & (-2) \cdot (-2) + 1 \cdot 2 & (-2) \cdot 3 + 1 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -6 & 6 & -1 \end{bmatrix}. \end{aligned}$$

Hence, $AB \neq BA$.

Example 3: If $[2x \quad 4] \begin{bmatrix} x \\ -8 \end{bmatrix} = 0$, find the positive value of x .

Solution: The given matrix equation is $AB = 0$, where A is a (1×2) matrix and B is a (2×1) matrix. So, AB is a (1×1) matrix.

So, 0 is a (1×1) matrix.

$$\begin{aligned} \therefore [2x \quad 4] \begin{bmatrix} x \\ -8 \end{bmatrix} &= [0] \\ \Rightarrow 2x^2 - 32 &= 0 \Rightarrow 2x^2 = 32 \\ \Rightarrow x^2 &= 16 \Rightarrow x = \sqrt{16} = 4. \end{aligned}$$

Hence, $x = 4$.

Properties of Matrix Multiplication

P 1: Matrix multiplication is not commutative in general.

Proof: Let A and B be two given matrices.

If AB exists then it is quite possible that BA may not exist.

For example, if A is a 3×2 matrix and B is a 2×2 matrix then clearly, AB exists but BA does not exist.

Similarly, if BA exists then AB may not exist.

For example, if A is a 2×3 matrix and B is a 2×2 matrix then clearly, BA exists but AB does not exist.

Further, if AB and BA both exist, they may not be comparable. For example, if A is a 2×3 matrix and B is a 3×2 matrix then clearly, AB as well as BA exists. But, AB is a 2×2 matrix while BA is a 3×3 matrix. Again, if AB and BA both exist and they are comparable, even then they may not be equal.

For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ then AB and BA are both defined and each one is a 2×2 matrix.

But, $AB = \begin{bmatrix} 1 & 5 \\ 1 & 8 \end{bmatrix}$ and $BA = \begin{bmatrix} 3 & 5 \\ 3 & 6 \end{bmatrix}$.

This shows that $AB \neq BA$.

Hence, in general, $AB \neq BA$.

REMARKS (i) When $AB = BA$, we say that A and B commute.

(ii) When $AB = -BA$, we say that A and B anticommute.

P 2. Associative

For any matrices A, B, C for which $(AB)C$ and $A(BC)$ both exist, we have

$$(AB)C = A(BC).$$

P 3: Distributive laws of multiplication over addition We have:

(i) $A \cdot (B + C) = (AB + AC)$

(ii) $(A + B) \cdot C = (AC + BC)$

P 4: The product of two nonzero matrices can be a zero matrix.

Example: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

Then, $A \neq O$ and $B \neq O$.

$$\text{And, } AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 2 + 1 \cdot 0 \\ 0 \cdot 1 + 2 \cdot 0 & 0 \cdot 2 + 2 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Left zero divisor If $AB = O$ and $A \neq O$ then A is called a left zero divisor of AB .

Right zero divisor If $AB = O$ and $B \neq O$ then B is called a right zero divisor of AB .

P 5: If A is a given square matrix and I is an identity matrix of the same order as A then we have $A \cdot I = I \cdot A = A$.

P 6: If A is a given square matrix and O is the null matrix of the same order as A then $O \cdot A = A \cdot O = O$.

Positive Integral Powers of a Square Matrix

Let A be a square matrix of order n . Then, we define:

$$\begin{aligned} A^2 &= A \cdot A; \\ A^3 &= A \cdot A \cdot A = A^2 \cdot A; \\ A^4 &= A \cdot A \cdot A \cdot A = A^3 \cdot A, \text{ and so on.} \\ \therefore A^n &= (A \cdot A \cdot A \cdot \dots \cdot n \text{ times}). \end{aligned}$$

Theorem 1: If A and B are square matrices of the same order then

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

Also, when $AB = BA$ then $(A + B)^2 = A^2 + 2AB + B^2$.

Proof: Let A and B be n -rowed square matrices.

Then, clearly, $(A + B)$ is a square matrix of order n .

So, $(A + B)^2$ is defined.

Now, $(A + B)^2 = (A + B) \cdot (A + B)$

$$\begin{aligned} &= A \cdot (A + B) + B \cdot (A + B) \quad [\text{by distributive law}] \\ &= AA + AB + BA + BB \quad [\text{by distributive law}] \\ &= A^2 + AB + BA + B^2. \end{aligned}$$

Hence, $(A + B)^2 = (A^2 + AB + BA + B^2)$.

Particular case When $AB = BA$

In this case, we have

$$(A + B)^2 = (A^2 + AB + AB + B^2) = (A^2 + 2AB + B^2) [\because BA = AB].$$

Theorem 2: If A and B are square matrices of the same order then

$$(A + B)(A - B) = A^2 - AB + BA - B^2.$$

Also, when $AB = BA$ then $(A + B)(A - B) = A^2 - B^2$.

Proof: We have:

$$\begin{aligned}(A + B) \cdot (A - B) &= A(A - B) + B(A - B) \text{ [by distributive law]} \\ &= AA - AB + BA - BB \text{ [} \because A(B - C) = AB - AC \text{]} \\ &= A^2 - AB + BA - B^2.\end{aligned}$$

Hence, $(A + B)(A - B) = A^2 - AB + BA - B^2$.

Particular case When $AB = BA$

In this case, $(A + B)(A - B) = (A^2 - B^2) [\because BA = AB]$.

Matrix Polynomial: Let $f(x) = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \dots + a_{m-1}x + a_m$ be a polynomial of degree m and let A be a square matrix of order n . Then, we define

$$f(A) = a_0A^m + a_1A^{m-1} + a_2A^{m-2} + \dots + a_{m-1}A + a_mI,$$

where I is a unit matrix of order n .

Solved Examples

Example 1: If $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix}$, find AB and BA whichever exists.

Solution: Here, A is a 2×2 matrix and B is a 2×3 matrix.
Clearly, the number of columns in A = number of rows in B .
 $\therefore AB$ exists and it is a 2×3 matrix.

$$\begin{aligned} AB &= \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 5 \cdot 3 + 4 \cdot 6 & 5 \cdot 5 + 4 \cdot 8 & 5 \cdot 1 + 4 \cdot 4 \\ 2 \cdot 3 + 3 \cdot 6 & 2 \cdot 5 + 3 \cdot 8 & 2 \cdot 1 + 3 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 15 + 24 & 25 + 32 & 5 + 16 \\ 6 + 18 & 10 + 24 & 2 + 12 \end{bmatrix} = \begin{bmatrix} 39 & 57 & 21 \\ 24 & 34 & 14 \end{bmatrix}. \end{aligned}$$

Again, B is a 2×3 matrix and A is a 2×2 matrix.

$\therefore$ number of columns in $B \neq$ number of rows in A .

So, BA does not exist.

Example 2: If $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$ then find AB and

BA .

Show that $AB \neq BA$.

Solution: Here, A is a 2×3 matrix and B is a 3×2 matrix

So, number of columns in A = number of rows in B .

$\therefore AB$ exists and it is a 2×2 matrix.

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 2 + (-1) \cdot 4 + 3 \cdot 1 & 2 \cdot 3 + (-1) \cdot (-2) + 3 \cdot 5 \\ -4 \cdot 2 + 5 \cdot 4 + 1 \cdot 1 & -4 \cdot 3 + 5 \cdot (-2) + 1 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 - 4 + 3 & 6 + 2 + 15 \\ -8 + 20 + 1 & -12 - 10 + 5 \end{bmatrix} = \begin{bmatrix} 3 & 23 \\ 13 & -17 \end{bmatrix}. \end{aligned}$$

Again, B is a 3×2 matrix and A is a 2×3 matrix.

So, number of columns in B = number of rows in A .

$\therefore BA$ exists and it is a 3×3 matrix.

$$\begin{aligned}
 BA &= \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 2 \cdot 2 + 3 \cdot (-4) & 2 \cdot (-1) + 3 \cdot 5 & 2 \cdot 3 + 3 \cdot 1 \\ 4 \cdot 2 + (-2) \cdot (-4) & 4 \cdot (-1) + (-2) \cdot 5 & 4 \cdot 3 + (-2) \cdot 1 \\ 1 \cdot 2 + 5 \cdot (-4) & 1 \cdot (-1) + 5 \cdot 5 & 1 \cdot 3 + 5 \cdot 1 \end{bmatrix} \\
 &= \begin{bmatrix} 4 - 12 & -2 + 15 & 6 + 3 \\ 8 + 8 & -4 - 10 & 12 - 2 \\ 2 - 20 & -1 + 25 & 3 + 5 \end{bmatrix} = \begin{bmatrix} -8 & 13 & 9 \\ 16 & -14 & 10 \\ -18 & 24 & 8 \end{bmatrix}.
 \end{aligned}$$

Clearly, $AB \neq BA$.

Example 3: If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$

then verify that $(AB)C = A(BC)$.

Solution: We have

$$\begin{aligned}
 AB &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 3 - 0 - 4 & 1 - 2 + 10 \\ 9 + 0 - 0 & 3 + 4 + 0 \\ -6 + 0 - 2 & -2 + 0 + 5 \end{bmatrix} = \begin{bmatrix} -1 & 9 \\ 9 & 7 \\ -8 & 3 \end{bmatrix} \\
 \Rightarrow (AB)C &= \begin{bmatrix} -1 & 9 \\ 9 & 7 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} -2 + 27 & -1 + 0 & 3 - 9 \\ 18 + 21 & 9 + 0 & -27 - 7 \\ -16 + 9 & -8 + 0 & 24 - 3 \end{bmatrix} = \begin{bmatrix} 25 & -1 & -6 \\ 39 & 9 & -34 \\ -7 & -8 & 21 \end{bmatrix}.
 \end{aligned}$$

$$\text{Also, } BC = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} 6+3 & 3+0 & -9-1 \\ 0+6 & 0+0 & 0-2 \\ -4+15 & -2+0 & 6-5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & -10 \\ 6 & 0 & -2 \\ 11 & -2 & 1 \end{bmatrix} \\
 \Rightarrow A(BC) &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 & -10 \\ 6 & 0 & -2 \\ 11 & -2 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 9-6+22 & 3-0-4 & -10+2+2 \\ 27+12+0 & 9+0-0 & -30-4+0 \\ -18+0+11 & -6+0-2 & 20-0+1 \end{bmatrix} \\
 &= \begin{bmatrix} 25 & -1 & -6 \\ 39 & 9 & -34 \\ -7 & -8 & 21 \end{bmatrix}.
 \end{aligned}$$

Hence, $(AB)C = A(BC)$.

Example 4: If $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix}$,

verify that $A(B + C) = (AB + AC)$.

Solution: We have

$$\begin{aligned}
 A(B + C) &= \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \left\{ \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix} \right\} \\
 &= \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 9 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 3 \cdot 9 + 2 \cdot 2 & 3 \cdot (-1) + 2 \cdot 2 & 3 \cdot (-1) + 2 \cdot 3 \\ 1 \cdot 9 + 0 \cdot 2 & 1 \cdot (-1) + 0 \cdot 2 & 1 \cdot (-1) + 0 \cdot 3 \end{bmatrix} \\
 &= \begin{bmatrix} 31 & 1 & 3 \\ 9 & -1 & -1 \end{bmatrix}.
 \end{aligned}$$

$$\text{Now, } AB = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} 3 \cdot 1 + 2 \cdot 0 & 3 \cdot (-2) + 2 \cdot 7 & 3 \cdot 5 + 2 \cdot 3 \\ 1 \cdot 1 + 0 \cdot 0 & 1 \cdot (-2) + 0 \cdot 7 & 1 \cdot 5 + 0 \cdot 3 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & 8 & 21 \\ 1 & -2 & 5 \end{bmatrix}.
 \end{aligned}$$

$$\text{And, } AC = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} 3 \cdot 8 + 2 \cdot 2 & 3 \cdot 1 + 2 \cdot (-5) & 3 \cdot (-6) + 2 \cdot 0 \\ 1 \cdot 8 + 0 \cdot 2 & 1 \cdot 1 + 0 \cdot (-5) & 1 \cdot (-6) + 0 \cdot 0 \end{bmatrix} \\
&= \begin{bmatrix} 28 & -7 & -18 \\ 8 & 1 & -6 \end{bmatrix}. \\
\therefore (AB + AC) &= \begin{bmatrix} 3 & 8 & 21 \\ 1 & -2 & 5 \end{bmatrix} + \begin{bmatrix} 28 & -7 & -18 \\ 8 & 1 & -6 \end{bmatrix} \\
&= \begin{bmatrix} 31 & 1 & 3 \\ 9 & -1 & -1 \end{bmatrix}.
\end{aligned}$$

Hence, $A(B + C) = AB + AC$.

Example 5: Give an example of two matrices A and B such that $A \neq O, B \neq O$ and $AB = BA = O$.

Solution: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Then,

$$\begin{aligned}
AB &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1-1 & -1+1 \\ 1-1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\text{and } BA &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Thus, $A \neq O, B \neq O$.

But, $AB = BA = O$.

Example 6: If $A = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$ and $B =$

$\begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}$, show that AB is a zero matrix if θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

Solution: We have

$$\begin{aligned}
 AB &= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix} \\
 &= \begin{bmatrix} \cos^2 \theta \cos^2 \phi + \cos \theta \sin \theta \cos \phi \sin \phi & \cos^2 \theta \cos \phi \sin \phi + \cos \theta \sin \theta \sin^2 \phi \\ \cos \theta \sin \theta \cos^2 \phi + \sin^2 \theta \cos \phi \sin \phi & \cos \theta \sin \theta \cos \phi \sin \phi + \sin^2 \theta \sin^2 \phi \end{bmatrix} \\
 &= \begin{bmatrix} \cos \theta & \cos \phi \cdot \cos(\theta - \phi) \\ \sin \theta & \cos \phi \cdot \cos(\theta - \phi) \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \phi \cdot \cos(\theta - \phi) \\ \sin \theta & \sin \phi \cdot \cos(\theta - \phi) \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \left[\because (\theta - \phi) \text{ being an odd multiple of } (\pi/2), \right. \\
 &\quad \left. \text{we have } \cos(\theta - \phi) = 0 \right].
 \end{aligned}$$

Example 7: If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A - 14I = O$.

Solution: We have

$$\begin{aligned}
 A^2 &= \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 3 \cdot 3 + (-5)(-4) & 3 \cdot (-5) + (-5) \cdot 2 \\ -4 \cdot 3 + 2 \cdot (-4) & -4 \cdot (-5) + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}; \\
 -5A &= (-5) \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix}; \\
 -14I &= (-14) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}. \\
 \therefore A^2 - 5A - 14I &= A^2 + (-5A) + (-14I) \\
 &= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} + \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix} + \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix} \\
 &= \begin{bmatrix} 29 + (-15) + (-14) & -25 + 25 + 0 \\ -20 + 20 + 0 & 24 + (-10) + (-14) \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
 \end{aligned}$$

Hence, $A^2 - 5A - 14I = O$.

Example 8: If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, find k so that $A^2 = 8A + kI$.

Solution: We have

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1-0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}; \\ (8A + kI) &= 8 \cdot \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} + k \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix} \\ \therefore A^2 = 8A + kI &\Rightarrow \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix} \\ \Rightarrow 8+k &= 1 \text{ and } 56+k = 49 \Rightarrow k = -7. \end{aligned}$$

Hence, $k = -7$.

Check Your Progress

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, find AB .

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.....

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2. Find $\det(A)$ for $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$.

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3. For $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$, calculate $\det(A)$ and state whether A is singular or non-singular.

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6.4 DETERMINANT

Determinant of a Square Matrix

Corresponding to each square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

there is associated an expression, called the determinant of A , denoted by $\det A$, or $|A|$, written as

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

A matrix is an arrangement of numbers and so it has no fixed value, while each determinant has a fixed value.

A determinant having n rows and n columns is known as a determinant of order n

The determinants of non-square matrices are not defined.

Value of a Determinant of Order 1: The value of a determinant of $a(1 \times 1)$ matrix $[a]$ is defined as $|a| = a$.

Value of a Determinant of Order 2: We define

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{21}a_{12}).$$

Example 1: Evaluate:

(i) $\begin{vmatrix} 6 & -3 \\ 7 & -2 \end{vmatrix}$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 6 & -3 \\ 7 & -2 \end{vmatrix} = 6(-2) - 7(-3) = -12 + 21 = 9.$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) - (x + 1)(x - 1)$
 $= (x^3 + 1) - (x^2 - 1) = (x^3 - x^2 + 2).$

Example 2: If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x .

Solution: We have $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

$$\begin{aligned} \Leftrightarrow (3x \times 4) - (-2) \times 7 &= (8 \times 4) - (6 \times 7) \\ \Leftrightarrow 12x + 14 &= 32 - 42 \\ \Leftrightarrow 12x + 14 &= -10 \\ \Leftrightarrow 12x &= -24 \Leftrightarrow x = -2. \end{aligned}$$

Hence, $x = -2$.

Example 3: If $\begin{vmatrix} x + 1 & x - 1 \\ x - 3 & x + 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, find the value of x .

Solution: We have $\begin{vmatrix} x + 1 & x - 1 \\ x - 3 & x + 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

$$\begin{aligned} \Leftrightarrow (x + 1)(x + 2) - (x - 3)(x - 1) &= (4 \times 3) - 1 \times (-1) \\ \Leftrightarrow (x^2 + 3x + 2) - (x^2 - 4x + 3) &= 12 + 1 \\ \Leftrightarrow 7x - 1 &= 13 \Leftrightarrow 7x = 14 \Leftrightarrow x = 2. \end{aligned}$$

Hence, $x = 2$.

Example 4: Show that $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix} = 1$.

Solution: We have $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix}$

$$\begin{aligned} &= (\sin 10^\circ)(\cos 80^\circ) - (\sin 80^\circ)(-\cos 10^\circ) \\ &= (\sin 10^\circ \cos 80^\circ + \cos 10^\circ \sin 80^\circ) \\ &= \sin (10^\circ + 80^\circ) [\because \sin A \cos B + \cos A \sin B = \sin (A + B)] \\ &= \sin 90^\circ = 1. \end{aligned}$$

Value of a Determinant of Order 3 or More: For finding the value of a determinant of order 3 or more, we need the following definitions.

Minor of a_{ij} in $|A|$: The minor of an element a_{ij} in $|A|$ is defined as the value of the determinant obtained by deleting the i th row and j th column of $|A|$, and it is denoted by M_{ij} .

Cofactor of a_{ij} in $|A|$: The cofactor C_{ij} of an element a_{ij} in $|A|$ is defined as $C_{ij} = (-1)^{i+j} \cdot M_{ij}$.

Example 1: Find the minors and cofactors of the elements of the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Solution: Let M_{ij} denote the minor of a_{ij} in Δ .

Now, a_{11} occurs in the 1st row and 1st column. So, in order to find the minor of a_{11} , we delete the 1st row and 1st column of Δ . The minor M_{11}

of a_{11} is given by $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}a_{33} - a_{32}a_{23})$.

Similarly, we have

$$\begin{aligned}
 M_{12} &= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{21}a_{33} - a_{31}a_{23}); \\
 M_{13} &= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (a_{21}a_{32} - a_{31}a_{22}); \\
 M_{21} &= \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = (a_{12}a_{33} - a_{32}a_{13}).
 \end{aligned}$$

Similarly, we may obtain the minor of each of the remaining elements.

Now, if we denote the cofactor of a_{ij} by C_{ij} then

$$\begin{aligned}
 C_{11} &= (-1)^{1+1} \cdot M_{11} = M_{11} = (a_{22}a_{33} - a_{32}a_{23}); \\
 C_{12} &= (-1)^{1+2} \cdot M_{12} = -M_{12} = (a_{31}a_{23} - a_{21}a_{33}); \\
 C_{13} &= (-1)^{1+3} \cdot M_{13} = M_{13} = (a_{21}a_{32} - a_{31}a_{22}); \\
 C_{21} &= (-1)^{2+1} \cdot M_{21} = -M_{21} = (a_{32}a_{13} - a_{12}a_{33}).
 \end{aligned}$$

Similarly, the cofactor of each of the remaining elements of Δ can be determined.

Example 2: Find the minor and cofactor of each element of $\Delta =$

$$\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}.$$

Solution: The minors of the elements of Δ are given by

$$\begin{aligned}
 M_{11} &= \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -12; M_{12} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = 2; M_{13} = \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix} = 23; \\
 M_{21} &= \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = -16; M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4; M_{23} = \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} = 14; \\
 M_{31} &= \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = -4; M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = -6; M_{33} = \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} = 11.
 \end{aligned}$$

So, the cofactors of the corresponding elements of Δ are

$$\begin{aligned}
 C_{11} &= (-1)^{1+1} \cdot M_{11} = M_{11} = -12; C_{12} = (-1)^{1+2} \cdot M_{12} = -M_{12} \\
 &= -2;
 \end{aligned}$$

$$\begin{aligned}
 C_{13} &= (-1)^{1+3} \cdot M_{13} = M_{13} = 23; C_{21} = (-1)^{2+1} \cdot M_{21} = -M_{21} = 16; \\
 C_{22} &= (-1)^{2+2} \cdot M_{22} = M_{22} = -4; C_{23} = (-1)^{2+3} \cdot M_{23} = -M_{23} = -14; \\
 C_{31} &= (-1)^{3+1} \cdot M_{31} = M_{31} = -4; C_{32} = (-1)^{3+2} \cdot M_{32} = -M_{32} = 6; \\
 C_{33} &= (-1)^{3+3} \cdot M_{33} = M_{33} = 11.
 \end{aligned}$$

Value of a Determinant

The value of a determinant is the sum of the products of elements of a row (or a column) with their corresponding cofactors.

We may expand a determinant by any arbitrarily chosen row or column.

Expansion of a Determinant

Expanding the given determinant by 1st row, we have

$$\begin{aligned}
 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11} \cdot (\text{its cofactor}) + a_{12} \cdot (\text{its cofactor}) + a_{13} \cdot (\text{its cofactor}) \\
 &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
 &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} [\because C_{12} = -M_{12}] \\
 &= a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
 &= a_{11} \cdot (a_{22}a_{33} - a_{32}a_{23}) - a_{12} \cdot (a_{21}a_{33} - a_{31}a_{23}) \\
 &\quad + a_{13} \cdot (a_{21}a_{32} - a_{31}a_{22}).
 \end{aligned}$$

We may expand it by any row or column.

Remark 1 If we expand a determinant by any row or column using minors, we keep in view the following symbols for a determinant of order three:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Remark 2 If a row or a column of a determinant consists of all zeros, the value of the determinant is zero.

Example 1: Evaluate

$$\Delta = \begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}.$$

Solution: Expanding the given determinant by 1st row, we get

$$\begin{aligned} \Delta &= 3 \cdot \begin{vmatrix} 2 & -3 \\ 1 & 7 \end{vmatrix} - 4 \cdot \begin{vmatrix} -6 & -3 \\ 8 & 7 \end{vmatrix} + 5 \cdot \begin{vmatrix} -6 & 2 \\ 8 & 1 \end{vmatrix} \\ &= 3 \cdot (14 + 3) - 4 \cdot (-42 + 24) + 5 \cdot (-6 - 16) = 13. \end{aligned}$$

Example 2: Expand the determinant $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Solution: Expanding by 1 st column, we get

$$\begin{aligned} \Delta &= a \cdot \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \cdot \begin{vmatrix} h & g \\ f & c \end{vmatrix} + g \cdot \begin{vmatrix} h & g \\ b & f \end{vmatrix} \\ &= a(bc - f^2) - h \cdot (ch - fg) + g \cdot (fh - bg) \\ &= abc - af^2 - ch^2 + fgh + fgh - bg^2 \\ &= (abc + 2fgh - af^2 - bg^2 - ch^2). \end{aligned}$$

Properties of Determinants

The properties of a determinant serve the purpose of a useful tool for finding its value. We will mention these properties and verify them for a third-order determinant.

Theorem 1: The value of a determinant remains unchanged if its rows and columns are interchanged.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by

interchanging the rows and columns of Δ .

Then,

$$\begin{aligned}
\Delta' &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \cdot \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\
&\quad [\text{expanded by 1st column}] \\
&= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - a_3b_2) \\
&= \Delta \quad [\text{expanded by 1st row}].
\end{aligned}$$

Note: If A is a square matrix then $|A'| = |A|$.

Theorem 2: If two rows or columns of a determinant are interchanged then the determinant retains its absolute value but its sign is changed.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by

interchanging any two rows, say 1st and 3rd rows, of Δ . Then,

$$\begin{aligned}
\Delta' &= \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \\
&= a_3 \cdot \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_1 \cdot \begin{vmatrix} b_3 & c_3 \\ b_2 & c_2 \end{vmatrix} \\
&\quad [\text{expanded by 1st column}] \\
&= a_3(b_2c_1 - b_1c_2) - a_2(b_3c_1 - b_1c_3) + a_1(b_3c_2 - b_2c_3) \\
&= -a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) - a_3(b_1c_2 - b_2c_1) \\
&= -[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)] \\
&= -\Delta \quad [\text{expanded by 1st column}].
\end{aligned}$$

Theorem 3: If any two rows or columns of a determinant are identical then its value is zero.

Proof: If we interchange the identical rows of the given determinant Δ then clearly there is no change in Δ . But, interchanging any two rows of a determinant changes its sign.

$$\therefore \Delta = -\Delta \Leftrightarrow 2\Delta = 0, \text{ i.e., } \Delta = 0.$$

Theorem 4: If each element of a row or a column of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the original determinant.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by multiplying each element of a row, say the second row of Δ , by k . Then,

$$\begin{aligned} \Delta' &= \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1 \cdot \begin{vmatrix} kb_2 & kc_2 \\ b_3 & c_3 \end{vmatrix} - ka_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ kb_2 & kc_2 \end{vmatrix} \\ &= a_1(kb_2c_3 - kb_3c_2) - ka_2(b_1c_3 - b_3c_1) + a_3(kb_1c_2 - kb_2c_1) \\ &= k[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)] = k\Delta. \end{aligned}$$

Important Note: For a square matrix A of order n , $|kA| = k^n \cdot |A|$.

Theorem 5: If any two rows or columns of a determinant are proportional then its value is zero.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix}$

$$\begin{aligned} &= k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \\ &= k \times 0 = 0 \quad [\text{by Theorem 4}] \\ &[\because \text{1st and 3rd rows are identical}]. \end{aligned}$$

Theorem 6: If each element of a row (or column) of a determinant is expressed as a sum of two or more terms then the determinant can be expressed as the sum of two or more determinants.
[by Theorem 4]

$$\Delta = \begin{vmatrix} a_1 + \alpha_1 & b_1 + \beta_1 & c_1 + \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Then, on expanding Δ by first row, we get

$$\begin{aligned} \Delta &= (a_1 + \alpha_1)(b_2c_3 - b_3c_2) - (b_1 + \beta_1)(a_2c_3 - a_3c_2) \\ &= [a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + \gamma_1(a_2b_3 - a_3b_2) - a_3b_2] \\ &\quad + [\alpha_1(b_2c_3 - b_3c_2) - \beta_1(a_2c_3 - a_3c_2) + \gamma_1(a_2b_3 - a_3b_2)] \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \end{aligned}$$

Theorem 7: If to any row or column of a determinant, a multiple of another row or column is added, the value of the determinant remains the same.

Let

$$\begin{aligned} \Delta &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \\ \Delta' &= \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \end{aligned}$$

Then,

$$\begin{aligned} \Delta' &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} [\because \text{2nd det. is 0, its 1st and 3rd} \\ &= \Delta. \end{aligned}$$

6.5 SUMMARY

So, in this first unit we have learned about

- Matrix multiplication
- Properties of matrix multiplication
- Determinant of a matrix
- Properties of determinant

6.6 GLOSSARY

Matrix Multiplication: An operation combining matrices when their dimensions are compatible.

Conformable Matrices: Matrices whose dimensions permit multiplication.

Scalar: A single number used to multiply a matrix.

Determinant: A scalar value associated with a square matrix.

Minor: The determinant obtained after deleting a row and a column.

Cofactor: A minor multiplied by its corresponding sign.

Adjoint: The transpose of the cofactor matrix.

Singular Matrix: A square matrix whose determinant is zero.

6.7 EXERCISES

1. Compute AB and BA , whenever exists when

(i) $A = \begin{bmatrix} 1 & 4 \\ 1 & 3 \\ 25 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 1 & 3 \\ 25 & 3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 5 & 2 \\ 13 & 4 & 2 \end{bmatrix}$

2. Verify that $A(B + C) = AB + AC$, when

(i) $A = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ 6 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & -2 \\ 2 & 4 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 1 \\ 3 & 5 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ 6 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$

3. If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, show that $A^2 = A$.

4. If $A = \begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$, show that $A^2 = I$.

5. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $(A^2 - 5A + 7I) = O$.

6. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the value of x and y such that $A^2 + xA = yA$.

7. If $\begin{vmatrix} 3x+1 & 3 \\ 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}$, find the value of x .

8. Find the determinant of the following matrices

$$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 6 & -2 & -4 \\ -1 & 0 & 4 \\ 1 & -2 & 5 \end{bmatrix}$$

6.8 SUGGESTED READINGS

1. Richard Bronson. Schaum's Outline of Matrix Operations. Tata McGraw Hill, 2011.
2. Hari Kishan. A Textbook of Matrices, Atlantic Publishers, 2023.
3. Shanti Narayan, P K Mittal. A Textbook of Matrices, S. Chand, 2010.

UNIT 7 RANK OF THE MATRIX

STRUCTURE

- 7.1. Introduction
- 7.2. Objectives
- 7.3. Sub-Matrix, Minor and Rank
- 7.4. Summary
- 7.5. Glossary
- 7.6. Exercises
- 7.7. Suggested Readings

7.1 INTRODUCTION

Welcome to the unit on the Rank of Matrix. In this module, we will delve into one of the fundamental concepts in linear algebra that plays a pivotal role in various mathematical and computational applications.

Understanding the Significance:

The rank of a matrix is a measure of its essential characteristics. It provides crucial insights into the properties of the matrix and its solutions. Whether in solving systems of linear equations, analysing transformations, or investigating the independence of vectors, the rank serves as a cornerstone concept.

7.2 OBJECTIVES

After completing this unit, students will be able to

- Define submatrix and minor of a matrix
- Understand the definition of rank
- Calculate rank of matrix with the help of elementary transformation

7.3 SUB-MATRIX, MINOR AND RANK

Sub-Matrix: A matrix, which is obtained by deleting some rows or some columns or both of matrix A , is called a sub-matrix of A . For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 0 & 11 \end{bmatrix}, \text{ then}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 6 & 7 \\ 10 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 0 & 11 \end{bmatrix}, \text{ etc. are all sub-}$$

matrices of the matrix A .

Particular Case. The matrix A is a sub-matrix of itself.

5.2. Minor of a Matrix

If any r rows and any r columns from an $m \times n$ matrix A are retained and the remaining $(m - r)$ rows and $(n - r)$ columns deleted, then the determinant of the remaining $r \times r$ sub-matrix of A is called a minor of A of order r . For example,

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix}_{5 \times 4}, \text{ then}$$

(i) The elements $a_{11}, a_{12}, a_{21}, a_{32}, a_{44}$, etc. are minors of A of order 1.

(i) The elements $a_{11}, a_{12}, a_{21}, a_{32}, a_{44}$, etc. are minors of A of order 1.

(ii) The determinants $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}, \begin{vmatrix} a_{13} & a_{14} \\ a_{53} & a_{54} \end{vmatrix},$

etc. are minors of A of order 2 .

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}, \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix},$ etc. are minors of

A of order 3.

(iv)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{vmatrix}, \text{ etc. are minors of } A \text{ of}$$

order 4 .

Note. There cannot be any minor of order higher than 4 in the above example.

5.3. Rank of a Matrix

A natural number r is called the rank of the matrix A if

- (i) There exists at least one non-zero minor of order r .
- (ii) Every minor of order $(r + 1)$, if any, vanishes. The rank of the matrix A is denoted by $\rho(A)$ or rank (A) .

Note 1. The rank of a zero (null) matrix of any order is taken to be zero.

Hence,

$$\rho(A) = 0 \text{ when } A \text{ is a zero matrix.}$$

Note 2. $\rho(A) \geq r$ when $A \neq 0$.

Note 3. $\rho(A) \leq r$ when every minor of order $(r + 1)$ vanishes.

Note 4. $\rho(A) \geq r$ when there exists a non-zero minor of order r .

Note 5. $\rho(A) \leq m$ when A is a $m \times n$ matrix such that $m \leq n$.

Note 6. $\rho(A) \leq n$ when A is a $m \times n$ matrix such that $n \leq m$.

Note 7. $\rho(A) = \rho(A')$ where A' is the transpose of A .

Note 8 . Rank of a matrix, whose every element is 1, is 1 .

Note 9. If A is a square matrix of order n such that $|A| \neq 0$, then $\rho(A) = n$.

Note 10. If I_n is an identity matrix of order n , then $|I_n| = 1 \neq 0$ and therefore $\rho(I_n) = n$.

Note 11. If A is a diagonal matrix of order n with nonzero diagonal elements, then $|A| \neq 0$ and therefore $\rho(A) = n$.

Note 12. The property (ii) of the number r in the definition of rank of a matrix implies that every minor of order $(r + 2)$ vanishes since every minor of order $(r + 2)$ can be expressed as the sum of the multiples of minors of order $(r + 1)$. It is simple to visualise that every minor of order higher than r will vanish. Thus, the rank of a matrix is the largest order of a non-vanishing minor of a matrix.

5.4. Nullity of a Matrix

Let A be a square matrix of order n . Then $n - \rho(A)$ is called the nullity of the matrix A and is denoted by $N(A)$. We know that the rank of a non-singular square matrix of order n is n and therefore its nullity is equal to zero.

Theorem 1. The rank of a matrix is equal to the rank of the transposed matrix.

Proof. We know that the transpose of a matrix is obtained by changing rows into columns and columns into rows. Also, by a property of determinants, we know that the value of a determinant remains unchanged if its rows are changed into columns and columns into rows. So, by transposing a matrix, the values of its minors do not change. Hence, $\rho(A) = \rho(A)$.

Theorem 2. $\rho(A^*) = \rho(A)$

Proof. We have proved in Theorem 1 that the rank of the transpose of a matrix is the same as the rank of the original matrix. So, to prove the theorem under consideration, what remains to prove for us is that the rank of the conjugate of a matrix is the same as the rank of the original matrix.

If A is real, then the result is trivial. If A is not real, then let us assume that the value of a minor of A with complex elements is $p + iq$. Then the value of the corresponding minor of the conjugate matrix of A is $p - iq$.

If $p + iq = 0$ then $p - iq = 0$

If $p + iq \neq 0$ then $p - iq \neq 0$

Hence,

$$\rho(\bar{A}) = \rho(A)$$

Also, $\rho(A) = \rho(A)$

In view of Eqs. (5.3) and (5.4), we have

$$\rho(A^*) = \rho(A)$$

Note. $\rho(A^*) = \rho(A^*) = \rho(A)$.

Theorem 3. If A is a non-zero column matrix and B is a non-zero row matrix, then show that $\rho(AB) = 1$.

Proof. Let $L = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ be a non-zero column matrix. Let $B =$

$[b_{11} b_{12} \dots b_{1n}]$ be a non-zero row matrix.

$\because A$ and B are both non-zero matrices.

$\therefore AB$ is also a non-zero matrix.

$\Rightarrow AB$ has at least one non-zero element.

$\Rightarrow AB$ will have at least one non-zero minor of order 1 .

$$\therefore \rho(AB) \geq 1$$

Check Your Progress

1. Find the rank of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ using elementary row operations.

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.....

2. Reduce $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$ to echelon form and find its rank.

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.....

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3. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ using elementary row operations.

.....

.....

.....

Now,

$$AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \dots & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & \dots & a_{21}b_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1}b_{11} & a_{n1}b_{12} & \dots & a_{n1}b_{1n} \end{bmatrix}$$

From the form of AB , it is clear that every minor of order 2 is zero.

Example 1: Find the rank of the matrix A , where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

Solution: We have

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \\ \therefore |A| &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} \quad \text{Operating } R_3 - R_1 \\ &= 0 \quad \because R_1 \text{ and } R_3 \text{ are identical} \\ \therefore \rho(A) &\neq 3 \end{aligned}$$

A minor of order 2 is $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

$$\therefore \rho(A) = 2$$

Example 2: Find the rank of the matrix A , where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$$

Solution:

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{vmatrix} \\ &= 1(10 - 6) + 2(12 - 8) + 3(4 - 10) \\ &= 4 + 8 - 18 \\ &= 6 \neq 0 \\ \therefore \rho(A) &= 3 \end{aligned}$$

Example 3: Find the rank of the matrix A , where

$$A = \begin{bmatrix} 8 & 0 & 0 & 1 \\ 1 & 0 & 8 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 8 & 1 & 8 \end{bmatrix}$$

Solution:

$$\begin{aligned} |A| &= \begin{vmatrix} 8 & 0 & 0 & 1 \\ 1 & 0 & 8 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 8 & 1 & 8 \end{vmatrix} \\ &= 8 \begin{vmatrix} 0 & 8 & 1 \\ 0 & 1 & 8 \\ 8 & 1 & 8 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 8 \\ 0 & 0 & 1 \\ 0 & 8 & 1 \end{vmatrix} \\ &= 8[8(64 - 1)] - [1(0 - 8)] \\ &= 4032 + 8 \\ &= 4040 \neq 0 \\ \therefore \rho(A) &= 4 \end{aligned}$$

Example 4: Find the rank of the matrix A , where

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Solution:

$$\begin{aligned} |A| &= \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{vmatrix} \\ &= 3 \begin{vmatrix} 6 & 1 & 1 & 8 \\ 4 & 2 & 2 & -1 \\ 10 & 3 & 3 & 7 \\ 16 & 4 & 4 & 15 \end{vmatrix} \quad \text{Taking 3 common from } c_3 \\ &= 3(0) \\ &= 0 \\ \therefore \rho(A) &\neq 4 \end{aligned}$$

It is easy to see that all the minors of order 3 are zero-valued.

$$\therefore \rho(A) \neq 3$$

There is a minor $\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix}$ of order 2 such that $\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 12 - 4 = 8 \neq 0$.

Hence, $\rho(A) = 2$

Example 5: Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 1 & 3 & 3 & 11 \end{bmatrix}$$

Solution: $\because$ The matrix A is of order 3×4

$$\therefore \rho(A) \leq 3$$

A minor of order 3 is

$$\begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 1 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & -6 \\ 0 & 4 & 0 \end{vmatrix} \quad \text{Operating } R_2 - R_1, R_3 - R_1$$

Example 6: Find the rank of the following matrix using elementary transformations:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Solution: We have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Operating $R_{21}(-1), R_{31}(-2)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Operating $R_{32}(-1)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The single minor of order 3 is zero.

A minor of order 2 is

$$\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 \neq 0$$

$$\therefore \rho(A) = 2$$

employing elementary transformations.

Example 6: Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$

employing elementary transformations.

Solution: We have

$$A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

Operating $R_{21}(-1), R_{31}(-1)$

$$A \sim \begin{bmatrix} 2 & 3 & 4 & -1 \\ 3 & -1 & -4 & 0 \\ -6 & 2 & 8 & 0 \end{bmatrix}$$

Operating $R_{32}(2)$

$$A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 3 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All the third order minors are zero but the second order

$$\text{minor } \begin{vmatrix} 4 & -1 \\ -4 & 0 \end{vmatrix} = -4 \neq 0$$
$$\therefore \rho(A) = 2$$

7.4 SUMMARY

So, in this first unit we have learned

- submatrix and minor of a matrix
- the definition of rank
- calculate rank of matrix with the help of minor and elementary transformation

7.5 GLOSSARY

Rank: The maximum number of linearly independent rows or columns of a matrix.

Elementary Row Operation: A basic operation used to transform a matrix.

Row Echelon Form: A matrix form with leading entries arranged progressively to the right.

Reduced Row Echelon Form: An echelon form with each leading entry equal to one and alone in its column.

Leading Entry: The first non-zero element in a non-zero row.

Zero Row: A row containing only zero elements.

Linear Independence: A condition in which no vector is a linear combination of the others.

Non-Singular Matrix: A square matrix having non-zero determinant.

7.6 EXERCISES

1. Find the rank of the following matrices

- $$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 6 & -2 & -4 \\ -1 & 0 & 4 \\ 1 & -2 & 5 \end{bmatrix}$$
2. If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ find the rank of A .
3. If $A = \begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$ find the rank of A by using elementary transformation.
4. If $B = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 5 & 2 \\ 13 & 4 & 2 \end{bmatrix}$ find the rank of B by using elementary transformation.

7.7 SUGGESTED READINGS

1. Richard Bronson. Schaum's Outline of Matrix Operations. Tata McGraw Hill, 2011.
2. Hari Kishan. A Textbook of Matrices, Atlantic Publishers, 2023.
3. Shanti Narayan, P K Mittal. A Textbook of Matrices, S. Chand, 2010.



UNIT 8 SYSTEM OF LINEAR EQUATIONS

STRUCTURE

- 8.1. Introduction
- 8.2. Objectives
- 8.3. System of Linear Equations
- 8.4. Summary
- 8.5. Glossary
- 8.6. Exercises
- 8.7. Suggested Readings

8.1 INTRODUCTION

Welcome to the unit on Systems of Linear Equations! In this module, we will embark on a journey to explore the fundamental concepts and techniques essential for understanding and solving systems of linear equations, which are prevalent in various fields ranging from mathematics and physics to engineering and economics.

Understanding the Significance:

A system of linear equations represents a collection of equations involving multiple variables, where each equation is linear. These systems serve as powerful tools for modeling real-world scenarios and finding solutions to complex problems. Whether it's determining the intersection of lines in geometry, analyzing electrical circuits, or optimizing production processes, systems of linear equations offer invaluable insights.

8.2 OBJECTIVES

After completing this unit, students will be able to

- Write system of linear equations in matrix form

- Differentiate homogeneous and non-homogeneous system
- Decide whether a non-homogeneous system has a solution or not
- Find the solution of system

8.3 SYSTEM OF LINEAR EQUATIONS

An equation of first degree in n unknowns $x_1, x_2, x_3, \dots, x_n$ is called a linear equation.

Thus,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

is a linear equation in n unknowns $x_1, x_2, x_3, \dots, x_n$ with coefficients $a_{11}, a_{12}, a_{13}, \dots, a_{1n}$ and b_1 as constants.

If $b_1 = 0$, then Eq. (6.1) takes the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0$$

and is called a homogeneous linear equation in n unknowns $x_1, x_2, x_3, \dots, x_n$.

Thus,

$2x + 3y + 4z = 20$ is an example of a non-homogeneous linear equation in 3 unknowns x, y and z whereas $x - y + z = 0$ is an example of a homogeneous linear equation in 3 unknowns x, y and z .

System of Linear Equations

Consider a system of m linear equations in n unknowns ($m > n, m = n$ or $m < n$) given below:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

In matrix notation, these equations can be put in the form

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

The matrix A is called the coefficient matrix. The matrix X is called the column matrix of n unknowns $x_1, x_2, x_3, \dots, x_n$. The matrix B is called the column matrix of m constants $b_1, b_2, \dots, b_m$.

The matrix $C = [A: B]$ obtained by placing the constant column matrix B to the right of the matrix A is called augmented matrix. Thus, the matrix

$$C = [A: B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \dots & \vdots & \dots \\ \dots & \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & \vdots & b_m \end{bmatrix}$$

is called the augmented matrix.

Any set of values of x_1, x_2, x_3, x_n which simultaneously satisfy the system of equation (A) is called the solution of the system (A). If the system has one or more solutions, it is called

consistent. If it has no solution, it is called inconsistent. A consistent system has either one solution or infinitely many solutions.

Example

(i) The system of equations

$$\begin{aligned}x_1 + x_2 &= 3 \\ 2x_1 + 3x_2 &= 8\end{aligned}$$

is consistent because it has a unique solution $x_1 = 1$,

$$x_2 = 2.$$

(ii) The system of equations

$$\begin{aligned}2x_1 + 3x_2 &= 5 \\ 6x_1 + 9x_2 &= 15\end{aligned}$$

is consistent because it has infinitely many solutions. Note that here the two equations are one and the same.

(iii) The system of equations

$$\begin{aligned}x_1 - 4x_2 &= 2 \\ 2x_1 - 8x_2 &= 5\end{aligned}$$

is inconsistent because it has no solution.

Non-singular or Regular System of Linear Equations

If we take $m = n$ in the system of equations (A), then we have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$
$$\therefore |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

$|A|$ is called the determinant of coefficients.

If $|A| \neq 0$, then the system of equations (when $m = n$) is called regular.

This system has a unique solution given by

$$\frac{x_1}{|A_1|} = \frac{x_2}{|A_2|} = \frac{x_3}{|A_3|} = \dots = \frac{x_n}{|A_n|} = \frac{1}{|A|} \text{ where } |A_1| \text{ represents the}$$

determinant obtained by replacing the r^{th} column of $|A|$ by the column of b 's.

This is known as Cramer's Rule for solving a system of n linear equation in n unknowns.

Note. If $|A| = 0$, then Cramer's Rule fails.

When $|A| \neq 0$, then A^{-1} exists. Hence, pre-multiplying the matrix equation

$$AX = B$$

by A^{-1} , we obtain,

$$\begin{aligned} A^{-1}(AX) &= A^{-1}B \\ \Rightarrow (A^{-1}A)X &= A^{-1}B \\ \Rightarrow IX &= A^{-1}B \\ \Rightarrow X &= A^{-1}B \end{aligned}$$

This gives the solution of the system of n equations in n unknowns when the system is non-singular (or regular).

Singular System of Linear Equations

If $|A| = 0$, the system of n equations in n unknowns is called singular.

This case will be dealt with later on.

System of Linear Equations, in General

Consider the system of linear equations (A)

It is simple to see that the rank of the augmented matrix C cannot be less than the rank of the coefficient matrix A because every sub-matrix of A is also a sub-matrix of C .

Let $\rho(A) = r$. Then, by a suitable sequence of elementary row operations, the matrix A can be reduced to an equivalent matrix in which each of the first r elements of the leading diagonal is 1 and every element below this diagonal and/or below the r^{th} row is zero. The matrix, so reduced, is said to be in Echelon Form.

If the same sequence of elementary operations is performed on the system of equations (A), then these will be transformed in the following form:

$$\begin{aligned} y_1 + \alpha_{12}y_2 + \alpha_{13}y_3 + \cdots + \alpha_{1n}y_n &= \beta_1 \\ y_2 + \alpha_{23}y_3 + \cdots + \alpha_{2n}y_n &= \beta_2 \\ y_r + \cdots + \alpha_{rn}y_n &= \beta_r \\ 0 &= \beta_{r+1} \\ \dots &= \dots \\ 0 &= \beta_m \end{aligned}$$

in which $y_1, y_2, \dots, y_n$ is some permutation of $x_1, x_2, \dots, x_n$.

Also, the coefficient matrix and the augmented matrix of the system of equations (B) will be equivalent to A and C respectively.

Now the following cases arise:

Case I. When $r = 1$, then the system of equations (B) becomes,

$$\left. \begin{aligned} y_1 + \alpha_{12}y_2 + \alpha_{13}y_3 + \cdots + \alpha_{1n}y_n &= \beta_1 \\ 0 &= \beta_1 \\ 0 &= \beta_3 \\ \vdots & \\ 0 &= \beta_m \end{aligned} \right\} \dots (C)$$

In this case, $\rho(C) = 2$ or 1 , since C has one column more than A .

When $\rho(C) = 2, \beta_2, \beta_3, \dots, \beta_m$ cannot all be zero. Hence, the equations are inconsistent and there will be no solution.

When $\rho(C) = 2, \beta_2, \beta_3, \dots, \beta_m$ will be all zero and the system of equations (B) will be equivalent to a single equation from which y_1 will be expressible in terms of $y_2, y_3, \dots, y_n$ which can have arbitrary values.

Case D. When $r = 2$, then the system of equations (B) becomes

$$\left. \begin{array}{l} y_1 + \alpha_{12}y_2 + \alpha_{13}y_3 + \dots + \alpha_{1n}y_n = \beta_1 \\ y_2 + \alpha_{23}y_3 + \dots + \alpha_{2n}y_n = \beta_2 \\ 0 = \beta_3 \\ 0 = \beta_4 \\ \vdots \\ 0 = \beta_m \end{array} \right\} \dots$$

In this case, $\rho(C) = 3$ or 2 .

When $\rho(C) = 3, \beta_3, \beta_4, \dots, \beta_m$ cannot all be zero. Hence, the equations are inconsistent and there will be no solution.

But when $\rho(C) = 2, \beta_3, \beta_4, \dots, \beta_m$ will be all zero and the equations will be equivalent to two independent equations from which y_1 and y_2 will be expressible in terms of $y_3, y_4, \dots, y_n$ which can have arbitrary values.

Similarly, when $r = 3$, then $\rho(C)$ must also be 3 in order that the equations may be consistent and in that case y_1, y_2, y_3 will be expressible in terms of $y_4, y_5, \dots, y_n$ which are arbitrary.

In general, the necessary and sufficient conditions that the equations (B) may be consistent is that $\rho(C) = \rho(A)$, i.e. if the coefficient matrix A and the augmented matrix C have the same rank and if each rank $= r$, the

equations will be equivalent to r equations from which r unknowns can be in further examples.

Check Your Progress

1. Solve: $2x + y = 7$ and $x - y = 1$.

.....

.....

.....

2. Determine whether the system $2x + 4y = 6$ and $x + 2y = 5$ is consistent or inconsistent.

.....

.....

.....

3. Solve by matrix method: $3x + 2y = 12$ and $x + y = 5$.

.....

.....

.....

Example 1: Solve, with the help of matrices, the simultaneous equations:

$$\begin{aligned}x + y + z &= 3 \\x + 2y + 3z &= 4 \\x + 4y + 9z &= 6\end{aligned}$$

Solution:

Coefficient Matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

$$\begin{aligned}\therefore |A| &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} \\ &= 1(18 - 12) + 1(9 - 3) + 1(4 - 2) \\ &= 6 - 6 + 2 \\ &= 2 \neq 0 \\ \therefore \rho(A) &= 3\end{aligned}$$

Augmented matrix $[A: B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 2 & 3 & : & 4 \\ 1 & 4 & 9 & : & 6 \end{bmatrix}$

Operating $R_2 - R_1, R_3 - R_1$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 3 & 8 & : & 3 \end{bmatrix}$$

Operating $R_3 - 3R_2$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 2 & : & 0 \end{bmatrix}$$

$$\therefore \rho[A: B] = 3$$

$$\therefore \rho[A] = \rho[A: B] = 3 \text{ (number of unknowns)}$$

$\therefore$ The given system of equations is consistent and has a unique solution.

Equivalent system of equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x + y + z &= 3 \\ y + 2z &= 1 \\ 2z &= 0 \end{aligned}$$

$$\Rightarrow x = 2, y = 1, z = 0$$

Example 2: Solve the system of equations by using matrix method:

$$\begin{aligned} 2x_1 + x_2 + 2x_3 + x_4 &= 6 \\ 6x_1 - 6x_2 + 6x_3 + 12x_4 &= 36 \\ 4x_1 + 3x_2 + 3x_3 - 3x_4 &= -1 \\ 2x_1 + 2x_2 - x_3 + x_4 &= 10 \end{aligned}$$

Solution:

Coefficient matrix $A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix}$

$$\therefore |A| \neq 0$$

$$\therefore \rho(A) = 4$$

$$\text{Augmented matrix } [A: B] = \begin{bmatrix} 2 & 1 & 2 & 1 & : & 6 \\ 6 & -6 & 6 & 12 & : & 36 \\ 4 & 3 & 3 & -3 & : & -1 \\ 2 & 2 & -1 & 1 & : & 10 \end{bmatrix}$$

Equivalent system of equations is

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -11 \\ 39 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 2x_1 + x_2 + 2x_3 + x_4 &= 6 \\ x_2 - x_4 &= -2 \\ -x_3 - 4x_4 &= -11 \\ 13x_4 &= 39 \end{aligned}$$

On solving, we get

$$x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$$

Example 3: Using matrix method, show that the equations

$$3x + 3y + 2z = 1$$

$$x + 2y = 4$$

$$10y + 3z = -2$$

$$2x - 3y - z = 5$$

are consistent and hence obtain the solutions for x, y and z .

Solution:

$$\text{Coefficient matrix } A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}$$

$$\therefore \text{Minor } \begin{vmatrix} 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{vmatrix} \neq 0$$

$$\therefore \rho(A) = 3$$

$$\therefore \rho[A: B] = 3$$

$$\therefore \rho[A: B] = \rho[A] = 3 \text{ (no. of variables)}$$

$\therefore$ The given system of equations is consistent and has a unique solution.

Equivalent system of equations is

$$\begin{bmatrix} 1 & 0 & -18 \\ 0 & 1 & 9 \\ 0 & 0 & 29 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 74 \\ -35 \\ -116 \\ 0 \end{bmatrix}$$

$$\Rightarrow x - 18z = 74$$

$$y + 9z = -35$$

$$29z = -116$$

$$\Rightarrow x = 2, y = 1, z = -4$$

Example 4: Show that the following system of equations is inconsistent:

$$x_1 - 2x_2 + x_3 - x_4 + 1 = 0$$

$$3x_1 - 2x_3 + 3x_4 + 4 = 0$$

$$5x_1 - 4x_2 + x_4 + 3 = 0$$

Solution:

Coefficient matrix $A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 3 & 0 & -2 & 3 \\ 5 & -4 & 0 & 1 \end{bmatrix}$

Augmented matrix $[A: B] = \begin{bmatrix} 1 & -2 & 1 & -1 & \vdots & -1 \\ 3 & 0 & -2 & 3 & \vdots & -4 \\ 5 & -4 & 0 & 1 & \vdots & -3 \end{bmatrix}$

Operating $R_2 - 3R_1, R_3 - 5R_1$

$$\begin{bmatrix} 1 & -2 & 1 & -1 & \vdots & -1 \\ 0 & 6 & -5 & 6 & \vdots & -1 \\ 0 & 6 & -5 & 6 & \vdots & 2 \end{bmatrix}$$

Operating $R_3 - R_1$

$$\begin{bmatrix} 1 & -2 & 1 & -1 & \vdots & -1 \\ 0 & 6 & -5 & 6 & \vdots & -1 \\ 0 & 0 & 0 & 0 & \vdots & 3 \end{bmatrix}$$

which is Echelon Form.

$$\text{Clearly, } \rho(A) = 2$$

$$\rho(A: B) = 3$$

$$\therefore \rho(A) \neq \rho(A: B)$$

Hence, the given system of equations is inconsistent.

Example 5: Investigate for consistency of the following equations and if possible, find the solutions.

$$\begin{aligned}4x - 2y + 6z &= 8 \\x + y - 3z &= -1 \\15x - 3y + 9z &= 21\end{aligned}$$

Solution:

Coefficient matrix $A = \begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}$

Operating R_{12}

$$- \begin{bmatrix} 1 & 1 & -3 & \vdots & -1 \\ 4 & -2 & 6 & \vdots & 8 \\ 15 & -3 & 9 & \vdots & 21 \end{bmatrix}$$

Operating $R_2 - 4R_1, R_3 - 15R_1$

$$- \begin{bmatrix} 1 & 1 & -3 & \vdots & -1 \\ 0 & -6 & 18 & \vdots & 12 \\ 0 & -18 & 54 & \vdots & 36 \end{bmatrix}$$

Operating $R_3 - 3R_2$

$$- \begin{bmatrix} 1 & 1 & -3 & \vdots & -1 \\ 0 & -6 & 18 & \vdots & 12 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

which is Echelon Form.

Clearly, $\rho(A) = 2 = \rho[A: B] < (\text{no. of variables})$

Hence, the given system of equations is consistent and has an infinite number of solutions.

Example 6. For what values of k , the equations $x + y + z = 1, 2x + y + 4z = k, 4x + y + 10z = k^2$ has a solution?

Solution. Here, we have

$$\begin{aligned}x + y + z &= 1 \\2x + y + 4z &= k \\4x + y + 10z &= k^2\end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

$AX = B$

$$\begin{aligned}C = [A:B] &= \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 1 & 4 & : & k \\ 4 & 1 & 10 & : & k^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & -3 & 6 & : & k^2-4 \end{bmatrix} \begin{matrix} \\ R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{matrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & 0 & 0 & : & k^2-3k+2 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-2 \\ k^2-3k+2 \end{bmatrix}\end{aligned}$$

If the given system has solutions, then $R(A) = R(C)$ But $R(A) = 2$

$$R(C) = 2 \text{ if } k^2 - 3k + 2 = 0 \Rightarrow (k-1)(k-2) = 0 \Rightarrow k = 1, k = 2$$

Case I. When $k = 1$, we have

$$x + y + z = 1 \quad (1)$$

and

$$-y + 2z = -1 \quad (2)$$

Let $z = \lambda$. Putting the value of $z = \lambda$ in (2), we have $y + 2\lambda = -1 \Rightarrow y = 2\lambda + 1$

Putting the values of y and z in (1), we have

$$x + (2\lambda + 1) + \lambda = 1 \Rightarrow x = -3$$

Hence solution is

$$\begin{aligned}x &= -3\lambda \\y &= 2\lambda + 1 \\z &= \lambda\end{aligned}$$

(λ is an arbitrary constant)

Case II. When $k = 2$, we have

$$x + y + z = 1 \quad (3)$$

and

$$-y + 2z = 0 \quad (4)$$

Putting the value of $z = c$ in (4), we have

$$-y + 2c = 0 \Rightarrow y = 2c$$

Putting the values of y and z in (1), we have

$$x + 2c + c = 1 \Rightarrow x = -3c + 1$$

Hence the solution is

$$x = 1 - 3c, y = 2c, z = c, \text{ where } c \text{ is an arbitrary constant.}$$

Example 7. Investigate the values of λ and μ so that the equations:

$$\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \lambda z &= \mu \end{aligned}$$

have (i) no solution

(ii) a unique solution

(iii) an infinite number of solutions.

Solution. Here, we have,

$$\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \lambda z &= \mu \end{aligned}$$

The above equations are written in the matrix form

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$C = [A: B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -\frac{15}{2} & -\frac{39}{2} \\ 0 & 0 & \lambda - 5 \end{bmatrix} \begin{array}{l} \\ \vdots -\frac{47}{2} \\ \vdots \mu - 9 \end{array} \left[\begin{array}{l} \\ R_2 \rightarrow R_2 - \frac{7}{2}R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \right]$$

(i) No solution. Rank $(A) \neq$ Rank (C)

$$\lambda - 5 = 0 \text{ or } \lambda = 5 \text{ and } \mu - 9 \neq 0$$

$$\Rightarrow \mu \neq 9$$

(ii) A unique solution. Rank $(A) = R(C) =$ Number of unknowns

$$\lambda - 5 \neq 0 \Rightarrow \lambda \neq 5 \text{ and } \mu \neq 9$$

(iii) An infinite number of solutions. Rank $(A) =$ Rank $(C) = 2$

$$\lambda - 5 = 0 \text{ and } \mu - 9 = 0$$

$$\lambda = 5 \text{ and } \mu = 9$$

8.4 SUMMARY

So, in this first unit we have learned about

- system of linear equations in matrix form
- Differentiate between homogeneous and non-homogeneous system
- whether a non-homogeneous system has a solution or not
- the calculation of the solution of system

8.5 GLOSSARY

Linear Equation: An equation in which variables have degree one.

System of Equations: A collection of two or more equations considered together.

Coefficient: A numerical factor multiplying a variable.

Constant: A fixed numerical quantity in an equation.

Solution: Values of variables satisfying all equations in a system.

Consistent System: A system having at least one solution.

Inconsistent System: A system having no solution.

Augmented Matrix: A matrix formed by adjoining the constants column to the coefficient matrix.

8.6 EXERCISES

Solve by matrix method:

(a) $x_1 - x_2 + x_3 = 2$

$$3x_1 - x_2 + 2x_3 = -6$$

$$3x_1 + x_2 + x_3 = -18$$

(b)

$$x + 2y + 3z = 14$$

$$3x + y + 2z = 11$$

$$2x + 3y + z = 11$$

(c)

$$2x - y + 3z = 9$$

$$x + y + z = 6$$

$$x - y + z = 2$$

(d)

$$x + y + z = 6$$

$$x + 2y + 3z = 14$$

$$x + 4y + 9z = 36$$

(c)

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 5x_2 + 6x_3 = 0$$

(f)

$$\begin{aligned}x + y + z &= 6 \\x - y + z &= 2 \\2x + y - z &= 1 \\x + y + z &= 3 \\x + 2y + 3z &= 4 \\x + 4y + 9z &= 6\end{aligned}$$

(g)

$$\begin{aligned}x + y + z &= 3 \\x + 2y + 3z &= 4 \\x + 4y + 9z &= 6\end{aligned}$$

(h)

$$\begin{aligned}x + 2y + 3z &= 4 \\2x + 3y + 8z &= 7 \\x - y + 9z &= 1\end{aligned}$$

(i)

$$\begin{aligned}4x + 2y + z + 34 &= 0 \\6x + 3y + 4z + 74 &= 0 \\2x + y + 4 &= 0\end{aligned}$$

(j)

$$\begin{aligned}x_1 + x_2 + x_3 + x_4 &= 1 \\2x_1 - x_2 + x_3 - 2x_4 &= 2 \\3x_1 + 2x_2 - x_3 - x_4 &= 3\end{aligned}$$

(k)

$$\begin{aligned}7x - 4y &= 12 \\-4x + 12y - 6z &= 0 \\-6y + 14z &= 0\end{aligned}$$

8.7 SUGGESTED READINGS

1. Richard Bronson. Schaum's Outline of Matrix Operations. Tata McGraw Hill, 2011.
2. Hari Kishan. A Textbook of Matrices, Atlantic Publishers, 2023.
3. Shanti Narayan, P K Mittal. A Textbook of Matrices, S. Chand, 2011.
4. R S Aggarwal, Senior Secondary School Mathematics, Bharti Bhawan, 2024.



BLOCK-III DIFFERENTIATION AND ITS APPLICATIONS TO ECONOMICS

UNIT 9 DERIVATIVE IN CALCULUS OF SINGLE VARIABLE

STRUCTURE

- 9.1. Introduction
- 9.2. Objectives
- 9.3. Limits
- 9.4. Continuity
- 9.5. The derivative and their notation
- 9.6. Rules of differentiation
- 9.7. Higher-order derivatives
- 9.8. Implicit Differentiation
- 9.9. List of some basic differentiation formulae
- 9.10. Summary
- 9.11. Glossary
- 9.12. Exercises
- 9.13. Suggested Readings

9.1 INTRODUCTION

Derivatives are a cornerstone of calculus, providing a way to measure how functions change. This unit delves into the foundational concepts of limits and continuity, which lead to the definition of derivatives. It covers the rules and techniques of differentiation, higher-order derivatives, and implicit differentiation. Mastering derivatives is essential for analyzing functions, solving optimization problems, and understanding rates of change in economics.

9.2 OBJECTIVES

After going through this unit, you should be able to

- Understand and compute limits of functions.
- Define and test the continuity of functions at a point and over an interval.
- Explain the concept of a derivative and its geometric and physical interpretations.
- Apply differentiation rules to find derivatives of algebraic, trigonometric, exponential, and logarithmic functions.
- Calculate higher-order derivatives.
- Perform implicit differentiation for functions defined implicitly.
- Utilize basic differentiation formulae to solve problems efficiently.

9.3 LIMITS

A real valued function $f(x)$ is said to have limit L at a point $x = a$ if

$\forall \epsilon > 0 \exists \delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Then we write $\lim_{x \rightarrow a} f(x) = L$.

i.e., we can set $f(x)$ as close as we want to l by making x sufficiently close to a .

If the functional values $f(x)$ of a function f draw closer to one and only one finite real number L for all values of x as x draws closer to a from both sides, but does not equal a , L is defined as the limit of $f(x)$ as x approaches a and is written

$$\lim_{x \rightarrow a} f(x) = L$$

Assuming that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist, the rules of limits are given below.

- 1 $\lim_{x \rightarrow a} k = k$ ($k = \text{a constant}$)
- 2 $\lim_{x \rightarrow a} x^n = a^n$ ($n = \text{a positive integer}$)
- 3 $\lim_{x \rightarrow a} kf(x) = k \lim_{x \rightarrow a} f(x)$ ($k = \text{a constant}$)
- 4 $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- 5 $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
- 6 $\lim_{x \rightarrow a} [f(x) \div g(x)] = \lim_{x \rightarrow a} f(x) \div \lim_{x \rightarrow a} g(x)$ [$\lim_{x \rightarrow a} g(x) \neq 0$]
- 7 $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$ ($n > 0$)

Example 1. In the absence of a graph, limits can be found by using the rules of limits enumerated above.

a) $\lim_{x \rightarrow 5} 9 = 9$ Rule 1

b) $\lim_{x \rightarrow 6} x^2 = (6)^2 = 36$ Rule 2

c) $\lim_{x \rightarrow 3} 2x^3 = 2 \lim_{x \rightarrow 3} x^3 = 2(3)^3 = 54$ Rules 2 and 3

d)

$$\begin{aligned} \lim_{x \rightarrow 2} (x^4 + 3x) &= \lim_{x \rightarrow 2} x^4 + 3 \lim_{x \rightarrow 2} x \\ &= (2)^4 + 3(2) = 22 \end{aligned}$$

e)

$$\begin{aligned} \lim_{x \rightarrow 4} [(x + 8)(x - 5)] &= \lim_{x \rightarrow 4} (x + 8) \cdot \lim_{x \rightarrow 4} (x - 5) \\ &= (4 + 8) \cdot (4 - 5) = -12 \end{aligned}$$

LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x \rightarrow a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Theorem: Let $p(x)$ and $q(x)$ be two polynomial functions. Let a be any real number. Then,

$$\lim_{x \rightarrow a} p(x) = p(a)$$

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

Proof: Consider the polynomial $p(x) = a_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$. Then

$$\begin{aligned} \lim_{x \rightarrow a} p(x) &= \lim_{x \rightarrow a} (c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0) \\ &= c_n a^n + c_{n-1} a^{n-1} + \dots + c_1 a + c_0 = p(a) \end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$, then

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

Example 2: Evaluate $\lim_{x \rightarrow 3} \frac{2x^2 - 3x + 1}{5x + 4}$

Solution: Here $f(x) = \frac{p(x)}{q(x)}$, where $p(x) = 2x^2 - 3x + 1$ and $q(x) = 5x + 4$ also $q(3) \neq 0$. Thus

$$\lim_{x \rightarrow 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{2 \cdot 3^2 - 3 \cdot 3 + 1}{5 \cdot 3 + 4} = \frac{10}{19}.$$

Calculating a limit when $\frac{f(x)}{g(x)}$ has the Indeterminate Form $\frac{0}{0}$.

- First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
- We then need to find a function that is equal to $h(x) = \frac{f(x)}{g(x)}$ for all $x \neq a$ over some interval containing a . To do this, we may need to try one or more of the following steps:

- If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
- If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
- If $\frac{f(x)}{g(x)}$ is a complex fraction, we begin by simplifying it.

➤ Last, we apply the limit laws.

Evaluating a Limit by Factoring and Canceling

Example 3: Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 3x}{2x^2 - 5x - 3}$.

Solution: $\lim_{x \rightarrow 3} \frac{x^2 - 3x}{2x^2 - 5x - 3} = \lim_{x \rightarrow 3} \frac{x(x-3)}{(x-3)(2x+1)} = \lim_{x \rightarrow 3} \frac{x}{2x+1} = \frac{3}{7}$.

Evaluating a Limit by Multiplying Conjugate

Example 4: Evaluate $\lim_{x \rightarrow -1} \frac{\sqrt{x+2}-1}{x+1}$.

Solution: $\lim_{x \rightarrow -1} \frac{\sqrt{x+2}-1}{x+1} = \lim_{x \rightarrow -1} \frac{\sqrt{x+2}-1}{x+1} \cdot \frac{\sqrt{x+2}+1}{\sqrt{x+2}+1} = \lim_{x \rightarrow -1} \frac{x+1}{(x+1)(\sqrt{x+2}+1)} = \lim_{x \rightarrow -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{2}$.

Evaluating a Limit When the Limit Laws do not apply

Example 5: Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right)$.

Solution: Both $\frac{1}{x}$ and $\frac{5}{x(x-5)}$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\begin{aligned}\lim_{x \rightarrow 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right) &= \lim_{x \rightarrow 0} \frac{x-5+5}{x(x-5)} \\ &= \lim_{x \rightarrow 0} \frac{x}{x(x-5)} = \lim_{x \rightarrow 0} \frac{1}{x-5} = -\frac{1}{5}.\end{aligned}$$

9.4 CONTINUITY

A continuous function is one which has no breaks in its curve. It can be drawn without lifting the pencil from the paper. A function f is continuous at $x = a$ if:

- 1 $f(x)$ is defined, i.e., exists, at $x = a$
- 2 $\lim_{x \rightarrow a} f(x)$ exists, and
- 3 $\lim_{x \rightarrow a} f(x) = f(a)$

All polynomial functions are continuous, as are all rational functions, except where undefined, i.e., where their denominators equal zero.

Examples:

1. $f(x) = 5x^2 - 8x + 9$ at $x = 3$

$$f(3) = 5(3)^2 - 8(3) + 9 = 30$$

$$\lim_{x \rightarrow 3} (5x^2 - 8x + 9) = 5(3)^2 - 8(3) + 9 = 30$$

$$\lim_{x \rightarrow 3} f(x) = 30 = f(3) \quad f(x) \text{ is continuous.}$$

2. $f(x) = \frac{x^2+3x+12}{x-3}$ at $x = 4$

$$f(4) = \frac{(4)^2 + 3(4) + 12}{4 - 3} = \frac{40}{1} = 40$$

$$\lim_{x \rightarrow 4} \frac{x^2 + 3x + 12}{x - 3} = 40$$

$$\lim_{x \rightarrow 4} f(x) = 40 = f(4) \quad f(x) \text{ is continuous.}$$

3. $f(x) = \frac{x-3}{x^2-9}$ at $x = 3$

$$f(3) = \frac{3-3}{(3)^2-9}$$

- 1) With the denominator equal to zero, $f(x)$ is not defined at $x = 3$ and so cannot be continuous at $x = 3$ even though the limit exists at $x = 3$. See steps 2 and 3.
- 2) $\lim_{x \rightarrow 3} \frac{x-3}{x^2-9} = \lim_{x \rightarrow 3} \frac{x-3}{(x+3)(x-3)} = \lim_{x \rightarrow 3} \frac{1}{x+3} = \frac{1}{6}$
- 3) $\lim_{x \rightarrow 3} f(x) = \frac{1}{6} \neq f(3)$. So $f(x)$ is discontinuous at $x = 3$.

9.5 THE DERIVATIVE AND THEIR NOTATION

Given a function $y = f(x)$, the derivative of the function f at x , written $f'(x)$ or dy/dx , is defined as

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ if the limit exists}$$

or,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

where $f'(x)$ is read "the derivative of f with respect to x " or " f prime of x ."

The derivative of a function $f'(x)$, or simply f' , is itself a function which measures both the slope and the instantaneous rate of change of the original function $f(x)$ at a given point.

The derivative of a function can be written in many different ways. If $y = f(x)$, the derivatives can be expressed as

$$f'(x) \quad y' \quad \frac{dy}{dx} \quad \frac{df}{dx} \quad \frac{d}{dx}[f(x)] \quad \text{or} \quad D_x[f(x)]$$

If $y = \phi(t)$, the derivative can be written

$$\phi'(t) \quad y' \quad \frac{dy}{dt} \quad \frac{d\phi}{dt} \quad \frac{d}{dt}[\phi(t)] \quad \text{or} \quad D_t[\phi(t)]$$

If the derivative of $y = f(x)$ is evaluated at $x = a$, proper notation includes $f'(a)$ and $\left.\frac{dy}{dx}\right|_a$.

9.6 RULES OF DIFFERENTIATION

9.6.1 The Constant Function Rule

The derivative of a constant function $f(x) = k$, where k is a constant, is zero.

Given $f(x) = k$,

$$\frac{d}{dx}f(x) = f'(x) = 0.$$

Example 1. If $f(x) = 5$. Then $\frac{d}{dx}f(x) = f'(x) = 0$.

9.6.2 The Linear Function Rule

The derivative of a linear function $f(x) = mx + b$ is equal to m , the coefficient of x . The derivative of a variable raised to the first power is always equal to the coefficient of the variable, while the derivative of a constant is simply zero.

Given $f(x) = mx + b$,

$$\frac{d}{dx}f(x) = f'(x) = m.$$

Example 2. Given $f(x) = 3x + 2$,

$$\frac{d}{dx}f(x) = f'(x) = 3.$$

Given $f(x) = 5 - \frac{1}{4}x$,

$$\frac{d}{dx}f(x) = f'(x) = -\frac{1}{4}.$$

Given $f(x) = 12x$,

$$\frac{d}{dx} f(x) = f'(x) = 12.$$

9.6.3 The Power Function Rule

The derivative of a power function $f(x) = kx^n$, where k is a constant and n is any real number, is equal to the coefficient k times the exponent n , multiplied by the variable x raised to the $n - 1$ power.

Given $f(x) = kx^n$

$$f'(x) = k \cdot n \cdot x^{n-1}$$

Example 3. Given $f(x) = 4x^3$, $f'(x) = 4 \cdot 3 \cdot x^{3-1} = 12x^2$

Given $f(x) = 5x^2$, $f'(x) = 5 \cdot 2 \cdot x^{2-1} = 10x$

Given $f(x) = x^4$, $f'(x) = 1 \cdot 4 \cdot x^{4-1} = 4x^3$

9.6.4 The Rules for Sums and Differences

The derivative of a sum of two functions $f(x) = g(x) + h(x)$, where $g(x)$ and $h(x)$ are both differentiable functions, is equal to the sum of the derivatives of the individual functions. Similarly, the derivative of the difference of two functions is equal to the difference of the derivatives of the two functions.

Given $f(x) = g(x) \pm h(x)$, $f'(x) = g'(x) \pm h'(x)$

Example 4. Given $f(x) = 12x^5 - 4x^4$, $f'(x) = 60x^4 - 16x^3$

Given $f(x) = 9x^2 + 2x - 3$, $f'(x) = 18x + 2$

9.6.5 The Product Rule

The derivative of a product $f(x) = g(x) \cdot h(x)$, where $g(x)$ and $h(x)$ are both differentiable functions, is equal to the first function multiplied by the derivative of the second plus the second function multiplied by the derivative of the first. Given $f(x) = g(x) \cdot h(x)$,

$$f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x)$$

Example 5. Given $f(x) = 3x^4(2x - 5)$, let $g(x) = 3x^4$ and $h(x) = 2x - 5$. Taking the individual derivatives, $g'(x) = 12x^3$ and $h'(x) = 2$. Then by substituting these values in the product-rule formula (3.3),

$$f'(x) = 3x^4(2) + (2x - 5)(12x^3)$$

and simplifying algebraically gives

$$f'(x) = 6x^4 + 24x^4 - 60x^3 = 30x^4 - 60x^3$$

9.6.6. The Quotient Rule

The derivative of a quotient $f(x) = g(x) \div h(x)$, where $g(x)$ and $h(x)$ are both differentiable functions and $h(x) \neq 0$, is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the denominator squared. Given $f(x) = g(x)/h(x)$,

$$f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

Example 6. Given

$$f(x) = \frac{5x^3}{4x + 3}$$

where $g(x) = 5x^3$ and $h(x) = 4x + 3$, we know that $g'(x) = 15x^2$ and $h'(x) = 4$. Substituting these values in the quotient-rule formula (3.4),

$$f'(x) = \frac{(4x + 3)(15x^2) - 5x^3(4)}{(4x + 3)^2}$$

Simplifying algebraically,

$$f'(x) = \frac{60x^3 + 45x^2 - 20x^3}{(4x + 3)^2} = \frac{40x^3 + 45x^2}{(4x + 3)^2} = \frac{5x^2(8x + 9)}{(4x + 3)^2}$$

9.6.7 The Generalized Power Function Rule

The derivative of a function raised to a power, $f(x) = [g(x)]^n$, where $g(x)$ is a differentiable function and n is any real number, is equal to the exponent n times the function $g(x)$ raised to the $n - 1$ power, multiplied in turn by the derivative of the function itself $g'(x)$. Given $f(x) = [g(x)]^n$,

$$f'(x) = n[g(x)]^{n-1} \cdot g'(x)$$

Example 7. Given $f(x) = (x^3 + 6)^5$, let $g(x) = x^3 + 6$, then $g'(x) = 3x^2$. Substituting these values in the generalized power function formula (3.5) gives

$$f'(x) = 5(x^3 + 6)^{5-1} \cdot 3x^2$$

Simplifying algebraically,

$$f'(x) = 5(x^3 + 6)^4 \cdot 3x^2 = 15x^2(x^3 + 6)^4$$

9.6.8 The Chain Rule

Given a composite function, also called a function of a function, in which y is a function of u and u in turn is a function of x , that is, $y = f(u)$ and $u = g(x)$, then $y = f[g(x)]$ and the derivative of y with respect to x is equal to the derivative of the first function with respect to u times the derivative of the second function with respect to x :

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 8. Consider the function $y = (5x^2 + 3)^4$. To use the chain rule, let $y = u^4$ and $u = 5x^2 + 3$. Then $dy/du = 4u^3$ and $du/dx = 10x$. Substitute these values in (3.6):

$$\frac{dy}{dx} = 4u^3 \cdot 10x = 40xu^3$$

Then to express the derivative in terms of a single variable, substitute $5x^2 + 3$ for u .

$$\frac{dy}{dx} = 40x(5x^2 + 3)^3$$

For more complicated functions, different combinations of the basic rules must be used.

Check Your Progress

1. Find dy/dx for $y = 3x^4 - 5x^2 + 7x - 2$.

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2. If $y = (2x + 1)(x^2 + 3)$, find dy/dx .

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3. Find the derivative of $y = (x^2 + 1)/x$, $x \neq 0$.

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9.7 HIGHER-ORDER DERIVATIVES

The second-order derivative, written $f''(x)$, measures the slope and the rate of change of the first derivative, just as the first derivative measures the slope and the rate of change of the original or primitive function. The

third-order derivative $f'''(x)$ measures the slope and rate of change of the second-order derivative, etc. Higher-order derivatives are found by applying the rules of differentiation to lower-order derivatives

Higher-order derivatives are found by successively applying the rules of differentiation to derivatives of the previous order. Thus, if $f(x) = 2x^4 + 5x^3 + 3x^2$,

$$\begin{aligned} f'(x) &= 8x^3 + 15x^2 + 6x \\ f''(x) &= 24x^2 + 30x + 6 \\ f'''(x) &= 48x + 30 \\ f^{(4)}(x) &= 48 \\ f^{(5)}(x) &= 0 \end{aligned}$$

9.8 IMPLICIT DIFFERENTIATION

Most explicit functions in introductory economics place the dependent variable to the left of the equal sign and the independent variable to the right. However, advanced economics courses often use implicit functions with variables and constants to the left of the equal sign. Not all implicit functions may be transformed to explicit by solving for the dependent variable in terms of the independent variable. Those not easily convertible may find the derivative using implicit differentiation.

Example 9. Samples of explicit and implicit functions include:

Explicit:

$$y = 4x \quad y = x^2 + 6x - 7 \quad y = \frac{x^4 - 9x^3}{x^2 - 13}$$

Implicit:

$$8x + 5y - 21 = 0 \quad 3x^2 - 8xy - 5y - 49 = 0 \quad 35x^3y^7 - 106 = 0$$

Example 10. Given $3x^4 - 7y^5 - 86 = 0$, the derivative dy/dx is found by means of implicit differentiation in two easy steps.

- 1 Differentiate both sides of the equation with respect to x while treating y as a function of x ,

$$\begin{aligned}\frac{d}{dx}(3x^4 - 7y^5 - 86) &= \frac{d}{dx}(0) \\ \frac{d}{dx}(3x^4) - \frac{d}{dx}(7y^5) - \frac{d}{dx}(86) &= \frac{d}{dx}(0)\end{aligned}$$

where $\frac{d}{dx}(3x^4) = 12x^3$, $\frac{d}{dx}(86) = 0$, $\frac{d}{dx}(0) = 0$. Using the generalized power function rule for $\frac{d}{dx}(7y^5)$ and noting that $\frac{d}{dx}(y) = \frac{dy}{dx}$, we get

$$\frac{d}{dx}(7y^5) = 7 \cdot 5 \cdot y^{5-1} \cdot \frac{d}{dx}(y) = 35y^4 \frac{dy}{dx}$$

Substitute the above values in (3.7).

$$12x^3 - 35y^4 \frac{dy}{dx} = 0$$

(2) Now simply solve (3.8) algebraically for dy/dx :

$$\begin{aligned}-35y^4 \frac{dy}{dx} &= -12x^3 \\ \frac{dy}{dx} &= \frac{12x^3}{35y^4}\end{aligned}$$

9.9 LIST OF SOME BASIC DIFFERENTIATION FORMULAE

a) $\frac{d}{dx}(x^n) = nx^{n-1}$

b) $\frac{d}{dx}(c) = 0$ (where c is a constant)

- c) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$
- d) $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- e) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
- f) $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$
- g) $\frac{d}{dx}(e^x) = e^x$
- h) $\frac{d}{dx}(\log_b(x)) = \frac{1}{x \cdot \ln(b)}$ (where b is the base of the logarithm)
- i) $\frac{d}{dx}(\sin(x)) = \cos(x)$
- j) $\frac{d}{dx}(\cos(x)) = -\sin(x)$
- k) $\frac{d}{dx}(\tan(x)) = \sec^2(x)$
- l) $\frac{d}{dx}(\csc(x)) = -\csc(x) \cot(x)$
- m) $\frac{d}{dx}(\sec(x)) = \sec(x) \tan(x)$
- n) $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
- o) $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$
- p) $\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$
- q) $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$
- r) $\frac{d}{dx}(\sinh(x)) = \cosh(x)$
- s) $\frac{d}{dx}(\cosh(x)) = \sinh(x)$

$$t) \frac{d}{dx}(\tanh(x)) = \operatorname{sech}^2(x)$$

$$u) \frac{d}{dx}(\operatorname{csch}(x)) = -\operatorname{csch}(x) \cdot \coth(x)$$

$$v) \frac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech}(x) \cdot \tanh(x)$$

$$w) \frac{d}{dx}(\coth(x)) = -\operatorname{csch}^2(x)$$

$$x) \frac{d}{dx}(\operatorname{arcsinh}(x)) = \frac{1}{\sqrt{x^2+1}}$$

$$y) \frac{d}{dx}(\operatorname{arccosh}(x)) = \frac{1}{\sqrt{x^2-1}}$$

$$z) \frac{d}{dx}(\operatorname{arctanh}(x)) = \frac{1}{1-x^2}$$

9.10 SUMMARY

In the present unit, we have covered the following concepts

- **Limits:** The fundamental building block of calculus, representing the value a function approaches as the input approaches a certain point.
- **Continuity:** A function is continuous at a point if its limit exists and equals the function's value at that point.
- **Derivative:** Measures the rate of change of a function. Defined as:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- **Rules of Differentiation:** Power rule, product rule, quotient rule, and chain rule are essential for computing derivatives.
- **Higher-order Derivatives:** The derivative of a derivative, representing more complex rates of change.
- **Implicit Differentiation:** Used to differentiate equations not explicitly solved for y .

- **Basic Differentiation Formulae:** Include derivatives of polynomials, trigonometric, exponential, and logarithmic functions.

9.11 GLOSSARY

Derivative: The instantaneous rate of change of a function.

Differentiation: The process of finding a derivative.

Limit: The value approached by a function as its argument approaches a specified value.

Continuity: A property where a function has no break at a point.

Differentiability: The property of having a derivative at a point.

Tangent: A straight line touching a curve at a point with the same local direction.

Rate of Change: The change in one quantity relative to change in another.

Chain Rule: A differentiation rule for composite functions.

9.12 EXERCISES

Compute the following limits

1. $\lim_{x \rightarrow 0} \frac{x^2 - 2}{x - 2}$
2. $\lim_{x \rightarrow 0} \frac{\sin x}{x}$
3. $\lim_{x \rightarrow \infty} \frac{5x}{x^2 + 1}$

4. Test for continuity at the given points:

a) $f(x) = \frac{x^2 - 1}{x - 1}$ at $x = 1$

b) $g(x) = |x - 3|$ at $x = 3$

5. Find the derivative of the following functions using basic differentiation rules:

a) $f(x) = 3x^5 - 7x^2 + 4$

b) $g(x) = \sin x \cdot e^x$

c) $h(x) = \frac{x^3}{x^2 + 1}$

6. Calculate higher-order derivatives for:
- a) $f(x) = x^4 - 2x^3 + x$
 - b) $g(x) = e^{-x} \sin x$
7. Use implicit differentiation to find $\frac{dy}{dx}$ for:
- a) $x^2 + y^2 = 25$
 - b) $x^3 + y^3 = 6xy$
8. Apply the chain rule to compute the derivative of:
- a) $f(x) = (3x^2 + 1)^5$
 - b) $g(x) = \tan(\ln x)$

9.13 SUGGESTED READINGS

1. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc., 2002.
2. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007.
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4. Gorakh Prasad, Differential Calculus, Pothishala Pvt. Ltd., Allahabad, 2016.



UNIT 10 APPLICATIONS OF DERIVATIVE IN ECONOMICS

STRUCTURE

- 10.1. Introduction
- 10.2. Objectives
- 10.3. Increasing and decreasing functions
- 10.4. Concavity and convexity
- 10.5. Relative extrema
- 10.6. Point of inflection
- 10.7. Optimization of functions
- 10.8. Marginal concepts
- 10.9. Optimizing economic functions
- 10.10. Relationship among total, marginal, and average concepts
- 10.11. Summary
- 10.12. Glossary
- 10.13. Exercises
- 10.14. Suggested Readings

10.1 INTRODUCTION

Derivatives play a crucial role in economics, providing tools to analyze and optimize functions representing cost, revenue, profit, and utility. By understanding the behavior of functions—whether they are increasing, decreasing, concave, or convex—economists can make informed decisions about production, pricing, and resource allocation. This unit explores how derivatives are used to determine relative extrema, points of inflection, and optimize economic functions. Additionally, the unit introduces marginal concepts and their relationship to total and average values, which are essential for understanding economic efficiency and profit maximization.

10.2 OBJECTIVES

After completing this unit, you will be able to

- Identify increasing and decreasing functions using the first derivative.
- Analyze the concavity and convexity of functions to determine their shape.
- Find relative maxima and minima (extrema) using critical points and the second derivative test.
- Locate points of inflection and understand their significance in economic models.
- Apply derivatives to optimize cost, revenue, and profit functions.
- Understand and compute marginal cost, marginal revenue, and marginal profit.
- Analyze the relationship between total, marginal, and average economic concepts to guide decision-making.

10.3 INCREASING AND DECREASING FUNCTIONS

A function $f(x)$ is said to be increasing (decreasing) at $x = a$ if in the immediate vicinity of the point $[a, f(a)]$ the graph of the function rises (falls) as it moves from left to right. Since the first derivative measures the rate of change and slope of a function, a positive first derivative at $x = a$ indicates the function is increasing at s ; a negative first derivative indicates it is decreasing. In short, as seen in Fig. 10.1,

$$\begin{aligned} f'(a) > 0: & \text{ increasing function at } x = a \\ f'(a) < 0: & \text{ decreasing function at } x = a \end{aligned}$$

A function that increases (or decreases) over its entire domain is called a monotonic function. It is said to increase (decrease) monotonically.



Figure 10.1: Increasing and decreasing function

10.4 CONCAVITY AND CONVEXITY

A function $f(x)$ is concave at $x = a$ if in some small region close to the point $[a, f(a)]$ the graph of the function lies completely below its tangent line. A function is convex at $x = a$ if in an area very close to $[a, f(a)]$ the graph of the function lies completely above its tangent line. A positive second derivative at $x = a$ denotes the function is convex at $x = a$; a negative second derivative at $x = a$ denotes the function is concave at a . The sign of the first derivative is irrelevant for concavity. In brief, as seen in Fig. 4-2 and Problems 4.1 to 4.4,

$$\begin{aligned} f''(a) > 0: & \quad f(x) \text{ is convex at } x = a \\ f''(a) < 0: & \quad f(x) \text{ is concave at } x = a \end{aligned}$$



Figure 10.2 Convex function



Figure 10.3 Concave function

10.5 RELATIVE EXTREMA

A relative extremum is a point at which a function is at a relative maximum or minimum. To be at a relative maximum or minimum at a point a , the function must be at a relative plateau, i.e., neither increasing nor decreasing at a . If the function is neither increasing nor decreasing at a , the first derivative of the function at a must equal zero or be undefined. A point in the domain of a function where the derivative equals zero or is undefined is called a critical point or value.

To distinguish mathematically between a relative maximum and minimum, the second-derivative test is used. Assuming $f'(a) = 0$,

- 1 If $f''(a) > 0$, indicating that the function is convex and the graph of the function lies completely above its tangent line at $x = a$, the function is at a relative minimum at $x = a$.
- 2 If $f''(a) < 0$, denoting that the function is concave and the graph of the function lies completely below its tangent line at $x = a$, the function is at a relative maximum at $x = a$.
- 3 If $f''(a) = 0$, the test is inconclusive.

For functions which are differentiable at all values of x , called differentiable or smooth functions, one need only consider cases where $f'(x) = 0$ in looking for critical points. To summarize,

$$\begin{array}{ll} f'(a) = 0 & f''(a) > 0: \text{ relative minimum at } x = a \\ f'(a) = 0 & f''(a) < 0: \text{ relative maximum at } x = a \end{array}$$

See Fig. 4-3 and Problems 4.5 and 4.6.



Figure 10.4 Relative extrema

10.6 POINT OF INFLECTION

An inflection point is a point on the graph where the function crosses its tangent line and changes from concave to convex or vice versa. Inflection points occur only where the second derivative equals zero or is undefined. The sign of the first derivative is immaterial. In sum, for an inflection point at a , as seen in Fig. 4-4 and Problems 4.6 and 4.7(c),

- 1 $f''(a) = 0$ or is undefined.
- 2 Concavity changes at $x = a$.
- 3 Graph crosses its tangent line at $x = a$.

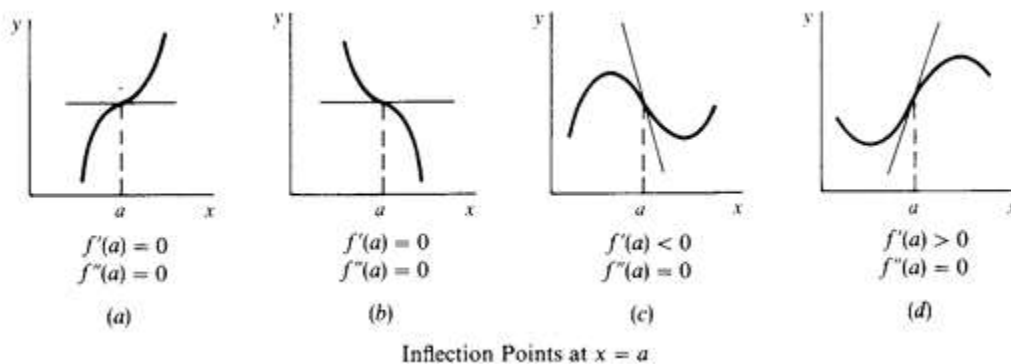


Figure 10.5 Point of Inflexion

10.7 OPTIMIZATION OF FUNCTIONS

Optimization is the process of finding the relative maximum or minimum of a function. Given the usual differentiable function,

1. Take the first derivative, set it equal to zero, and solve for the critical point(s). This step represents a necessary condition known as the first-order condition. It identifies all the points at which the function

is neither increasing nor decreasing, but at a plateau. All such points are candidates for a possible relative maximum or minimum.

2. Take the second derivative, evaluate it at the critical point(s), and check the sign(s). If at a critical point a , $f''(a) < 0$, the function is concave at a , and hence at a relative maximum.
 $f''(a) > 0$, the function is convex at a , and hence at a relative minimum.
 $f''(a) = 0$, the test is inconclusive. See Section 4.6. Assuming the necessary first-order condition is met, this step, known as the second-order derivative test, or simply the second-order condition, represents a sufficiency condition. In sum,

Relative maximum	Relative minimum
$f'(a) = 0$	$f'(a) = 0$
$f''(a) < 0$	$f''(a) > 0$

Note that if the function is strictly concave (convex), there will be only one maximum (minimum), called a global maximum (minimum). See Example 1 and Problems 4.7 to 4.9 .

Example 1. Optimize $f(x) = 2x^3 - 30x^2 + 126x + 59$.

- a) Find the critical points by taking the first derivative, setting it equal to zero, and solving for x .

$$\begin{aligned} f'(x) &= 6x^2 - 60x + 126 = 0 \\ 6(x - 3)(x - 7) &= 0 \\ x = 3 \quad x = 7 &\text{ critical points} \end{aligned}$$

- (b) Test for concavity by taking the second derivative, evaluating it at the critical points, and checking the signs to distinguish between a relative maximum and minimum.

$$f''(x) = 12x - 60$$

$$f''(3) = 12(3) - 60 = -24 < 0 \text{ concave, relative maximum}$$

$$f''(7) = 12(7) - 60 = 24 > 0 \text{ convex, relative minimum}$$

The function is maximized at $x = 3$ and minimized at $x = 7$.

10.8 MARGINAL CONCEPTS

Marginal cost in economics is defined as the change in total cost incurred from the production of an additional unit. Marginal revenue is defined as the change in total revenue brought about by the sale of an extra good. Since total cost (TC) and total revenue (TR) are both functions of the level of output (Q), marginal cost (MC) and marginal revenue (MR) can each be expressed mathematically as derivatives of their respective total functions.

Thus,

$$\text{if } TC = TC(Q), \text{ then } MC = \frac{dTC}{dQ}$$

$$\text{and if } TR = TR(Q), \text{ then } MR = \frac{dTR}{dQ}$$

In short, the marginal concept of any economic function can be expressed as the derivative of its total function.

Example 1.

$$1 \quad \text{If } TR = 75Q - 4Q^2, \text{ then } MR = dTR/dQ = 75 - 8Q.$$

$$2 \quad \text{If } TC = Q^2 + 7Q + 23, \text{ then } MC = dTC/dQ = 2Q + 7.$$

Example 2. Given the demand function $P = 30 - 2Q$, the marginal revenue function can be found by first finding the total revenue function and then taking the derivative of that function with respect to Q . Thus,

$$TR = PQ = (30 - 2Q)Q = 30Q - 2Q^2$$

Then

$$MR = \frac{dTR}{dQ} = 30 - 4Q$$

If $Q = 4$, $MR = 30 - 4(4) = 14$; if $Q = 5$, $MR = 30 - 4(5) = 10$.

Check Your Progress

1. If the total cost function is $C(q) = 100 + 20q + 2q^2$, find the marginal cost at $q = 5$.

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2. If the total revenue function is $R(q) = 50q - q^2$, find the marginal revenue at $q = 10$.

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3. Given $\pi(q) = 60q - 2q^2 - 100$, find the output level that maximises profit and verify the maximum using the second derivative.

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10.9 OPTIMIZING ECONOMIC FUNCTIONS

The economist is frequently called upon to help a firm maximize profits and levels of physical output and productivity, as well as to minimize costs, levels of pollution, and the use of scarce natural resources. This is done with the help of techniques developed earlier.

Example 1. Maximize profits π for a firm, given total revenue $R = 4000Q - 33Q^2$ and total cost $C = 2Q^3 - 3Q^2 + 400Q + 5000$, assuming $Q > 0$.

a) Set up the profit function: $\pi = R - C$.

$$\begin{aligned}\pi &= 4000Q - 33Q^2 - (2Q^3 - 3Q^2 + 400Q + 5000) \\ &= -2Q^3 - 30Q^2 + 3600Q - 5000\end{aligned}$$

b) Take the first derivative, set it equal to zero, and solve for Q to find the critical points.

$$\begin{aligned}\pi' &= -6Q^2 - 60Q + 3600 = 0 \\ &= -6(Q^2 + 10Q - 600) = 0 \\ &= -6(Q + 30)(Q - 20) = 0 \\ Q &= -30 \quad Q = 20 \text{ critical points}\end{aligned}$$

c) Take the second derivative; evaluate it at the positive critical point and ignore the negative critical point, which has no economic significance and will prove mathematically to be a relative minimum. Then check the sign for concavity to be sure of a relative maximum.

$$\begin{aligned}\pi'' &= -12Q - 60 \\ \pi''(20) &= -12(20) - 60 = -300 < 0 \text{ concave, relative maximum}\end{aligned}$$

Profit is maximized at $Q = 20$ where

$$\pi(20) = -2(20)^3 - 30(20)^2 + 3600(20) - 5000 = 39,000.$$

10.10 RELATIONSHIP AMONG TOTAL, MARGINAL, AND AVERAGE CONCEPTS

A total product (TP) curve of an input is derived from a production function by allowing the amounts of one input (say, capital) to vary while holding the other inputs (labor and land) constant. A graph showing the relationship between the total, average, and marginal products of an input can easily be sketched by using now familiar methods, as demonstrated in the following example.

Example 1. Given $TP = 90K^2 - K^3$, the relationship among the total, average, and marginal products can be illustrated graphically as follows.

- 1 Test the first-order condition to find the critical values.

$$TP' = 180K - 3K^2 = 0$$

$$3K(60 - K) = 0$$

$$K = 0 \quad K = 60 \quad \text{critical values}$$

Check the second-order conditions.

$$TP'' = 180 - 6K$$

$$TP''(0) = 180 > 0$$

$$TP''(60) = -180 < 0 \quad \text{convex, relative minimum}$$

concave, relative maximum

Check for inflection points.

Since, at $K = 30$, $TP'' = 0$ and concavity changes, there is an inflection point at $K = 30$.

2. Find and maximize the average product of capital AP_K .

$$AP_K = \frac{TP}{K} = 90K - K^2$$

$$AP'_K = 90 - 2K = 0$$

$$K = 45 \quad \text{critical value}$$

$$AP'_K = -2 < 0 \quad \text{concave, relative maximum}$$

10.11 SUMMARY

In this unit, we have covered the following concepts

- **Increasing and Decreasing Functions:** A function is increasing when its derivative is positive and decreasing when the derivative is negative.
- **Concavity and Convexity:**

Concave functions have a negative second derivative, while convex functions have a positive second derivative.
- **Relative Extrema:** Points where the function reaches local maxima or minima, identified by setting the first derivative to zero and using the second derivative test.
- **Point of Inflection:** A point where the concavity of a function changes, found by setting the second derivative equal to zero.
- **Optimization of Functions:** Derivatives help maximize profit, minimize cost, and optimize production functions.
- **Marginal Concepts:** Marginal cost, revenue, and profit represent the rate of change of total cost, revenue, and profit, respectively.
- **Relationship among Total, Marginal, and Average:**

Marginal values are the derivatives of total functions.

Average functions are obtained by dividing the total by the quantity produced.

10.12 GLOSSARY

Marginal Cost: The change in total cost caused by a small change in output.

Marginal Revenue: The change in total revenue caused by a small change in output.

Marginal Product: The additional output resulting from one more unit of an input.

Profit Function: A function representing revenue minus cost.

Maximum: A value greater than nearby values of a function.

Minimum: A value smaller than nearby values of a function.

Optimization: The process of finding maximum or minimum values.

Elasticity: A measure of responsiveness of one economic variable to another.

10.13 EXERCISES

- 1 Determine whether the following functions are increasing or decreasing:
 - a) $f(x) = 4x^3 - 9x^2 + 6x - 1$
 - b) $g(x) = e^{-x} \cdot (x + 2)$
- 2 Analyze concavity and points of inflection:
 - a) $f(x) = x^4 - 4x^3 + 6x^2$
 - b) $g(x) = \ln(x^2 + 1)$
- 3 Find the relative extrema of the following functions:
 - a) $f(x) = x^3 - 6x^2 + 9x + 5$
 - b) $g(x) = 5x - \frac{1}{x}$
- 4 Locate the points of inflection for:
 - a) $h(x) = x^5 - 5x^3 + 4x$
 - b) $k(x) = e^x(x - 3)$
- 5 Optimize the following economic functions:
 - a) Profit function $P(x) = -2x^2 + 10x - 12$
 - b) Cost function $C(x) = x^3 - 3x^2 + 7x + 5$

- 6 Compute the marginal concepts for the following total functions:
- a) Total revenue $R(x) = 10x - 0.5x^2$
 - b) Total cost $C(x) = 3x^2 + 5x + 20$
- 7 Establish the relationship between total, marginal, and average for the following:
- a) $T(x) = 12x^2 + 3x + 15$
 - b) $T(x) = x^3 - 4x^2 + 6x + 9$.

10.14 SUGGESTED READINGS

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UNIT 11 DIFFERENTIATION OF MULTIVARIABLE FUNCTIONS

STRUCTURE

- 11.1. Introduction
- 11.2. Objectives
- 11.3. Partial Derivatives
- 11.4. Optimization of multivariable functions
- 11.5. Constrained optimization with Lagrange multipliers
- 11.6. Total and partials differentials and total derivative
- 11.7. Rules for implicit and inverse function
- 11.8. Summary
- 11.9. Glossary
- 11.10. Exercises
- 11.11. Suggested Readings

11.1 INTRODUCTION

Multivariable functions extend the concept of single-variable calculus to functions that depend on two or more variables. These functions are essential in economics, where multiple factors often influence outcomes simultaneously. For example, production levels may depend on both labor and capital, while demand may vary with price and income. Understanding how to analyze and optimize such functions is crucial for economic modeling, forecasting, and decision-making. This unit explores partial derivatives, optimization techniques, constrained maximization with Lagrange multipliers, and the application of total differentials and implicit functions.

11.2 OBJECTIVES

By the end of this unit, you will be able to:

- Understand and compute partial derivatives of multivariable functions.
- Apply optimization techniques to functions with multiple variables.
- Perform constrained optimization using Lagrange multipliers.
- Utilize total and partial differentials to analyze small changes in economic functions.
- Apply implicit differentiation and inverse function rules to solve complex economic models.

11.3 PARTIAL DERIVATIVES

Let $z = f(x, y)$ is a function of two variables x and y then may have three cases

- I. Only variable x is varying while variable y is fixed.
- II. Only variable y is varying while variable x is fixed.
- III. Both the variables x and y varying simultaneously.

In first case, if we differentiate z with usual rules with respect to x regarding y as a constant, then this derivative of z is called partial derivative of z with respect to z and denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x and defined by

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Similarly, partial derivative of z with respect to y and denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y and defined by

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

Higher Order Partial Derivatives: Let $z = f(x, y)$ then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ being the functions of x and y can be further differentiated partially w.r.t. x and y . These derivatives are called second order partial derivatives of z and denoted by

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

Note that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ also we write $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial y^2} = t, \frac{\partial^2 z}{\partial x \partial y} = s$.

Note: Let $f(x, y, z)$ is a function of three variables x, y and z . The notation f_{zyzyx} is shorthand for $\left(\left(\left((f_z)_y \right)_z \right)_y \right)_x$, so we differentiate with respect to z , then with respect to y , then z , then y , then x . That is, we read left to right.

It's worth pointing out that the order is different in the other notation:

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial y} \frac{\partial f}{\partial z} = \frac{\partial^5 f}{\underbrace{\frac{\partial x}{\partial x} \frac{\partial y}{\partial y} \frac{\partial z}{\partial z} \frac{\partial y}{\partial y} \frac{\partial z}{\partial z}}_{5^{th} \ 4^{th} \ 3^{rd} \ 2^{nd} \ 1^{st}}}$$

Example 1. If $f(x, y) = \sin(xy) + 3xy$ then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution: $\frac{\partial f}{\partial x} = \cos(xy)y + 3y$ $\frac{\partial f}{\partial y} = \cos(xy)x + 3x$.

Example 2. If $z = \frac{x^2}{a^2} + \frac{y^2}{b^2} + 10$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: Given that,

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} + 10$$

Differentiate z partially w.r.t. x (treating y as a constant) we get

$$\frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

Differentiate z partially w.r.t. y (treating x as a constant) we get

$$\frac{\partial z}{\partial y} = \frac{2y}{b^2}.$$

Example 3. If $f(x, y) = \frac{xy}{x+y^3}$ then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution: Differentiate f partially w.r.t. x (treating y as a constant) we get

$$f_x(x, y) = y \frac{\partial}{\partial x} \frac{x}{x+y^3} = y \frac{(x+y^3) - x}{(x+y^3)^2} = \frac{y^4}{(x+y^3)^2}.$$

Differentiate z partially w.r.t. y (treating x as a constant) we get

$$f_y(x, y) = x \frac{\partial}{\partial y} \frac{y}{x+y^3} = x \frac{(x+y^3) - 3y^2}{(x+y^3)^2} = \frac{x(x-2y^3)}{(x+y^3)^2}.$$

Example 4. If $f(x, y) = x \log(x + y^3)$ then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution: Differentiate f partially w.r.t. x (treating y as a constant) we get

$$f_x(x, y) = \ln(x + y^3) + x \frac{\partial}{\partial x} \ln(x + y^3) = \ln(x + y^3) + \frac{x}{x + y^3},$$

Differentiate f partially w.r.t. y (treating x as a constant) we get

$$f_y(x, y) = x \frac{\partial}{\partial y} \ln(x + y^3) = \frac{3xy^2}{x + y^3}.$$

Example 4. Find all first order partial derivatives of the function

$$w = f(x, y, z) = xyz - xe^x + x \sin y.$$

Solution: Given that

$$w = f(x, y, z) = xyz - xe^x + x \sin y.$$

w is a function of three variables x, y and z , therefore

$$f_x(x, y) = \frac{\partial}{\partial x} (xyz - xe^x + x \sin y) = yz - (e^x + xe^x) + \sin y$$

$$f_y(x, y) = \frac{\partial}{\partial y} (xyz - xe^x + x \sin y) = xz + x \cos y$$

$$f_z(x, y) = \frac{\partial}{\partial z} (xyz - xe^x + x \sin y) = xy.$$

Example 5. Find all the second order partial derivatives of $f(x, y) = \sin(x)y^2$.

Solution: First, we find both first order partial derivatives:

$$\frac{\partial}{\partial x} (\sin(x)y^2) = \cos(x)y^2$$

$$\frac{\partial}{\partial y}(\sin(x) y^2) = 2 \sin(x) y$$

Now we compute second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (\cos(x) y^2) = -\sin(x) y^2,$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (2 \sin(x) y) = 2 \sin(x),$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (\cos(x) y^2) = 2 y \cos(x),$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (2 \sin(x) y) = 2 y \cos(x).$$

11.4 OPTIMIZATION OF MULTIVARIABLE FUNCTIONS

Definition 11.4.1. Relative Extrema of a Two-Variable Function.

Let $z = f(x, y)$ be a function defined on a region $R \subseteq \mathbb{R}^2$ containing the point (x_0, y_0) .

Then f has a **relative maximum** at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all points (x, y) that are sufficiently close to (x_0, y_0) . The number $f(x_0, y_0)$ is the relative maximum **value**.

Then f has a **relative minimum** at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all points (x, y) that are sufficiently close to (x_0, y_0) . The number $f(x_0, y_0)$ is the relative minimum **value**.

Definition 11.4.2 Absolute Extrema of a Two-Variable Function.

Let $z = f(x, y)$ be a function defined on a region $R \subseteq R^2$ containing the point (x_0, y_0) .

Then f has an **absolute maximum** at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all points $(x, y) \in R$. The number $f(x_0, y_0)$ is the absolute maximum **value**.

Then f has an **absolute minimum** at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all points $(x, y) \in R$. The number $f(x_0, y_0)$ is the relative minimum **value**.

Definition 11.4.3 Critical Point of a Two-Variable Function.

Let $z = f(x, y)$ be a function defined on a region $R \subseteq R^2$ containing the point (x_0, y_0) .

The point (x_0, y_0) in R is a critical point of f if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or at least one of the derivatives $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Theorem 11.4.4. Second Derivative Test for Two-Variable Functions.

Suppose that the second partial derivatives of $f(x, y)$ are continuous near (x_0, y_0) , and $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. We denote by D the **discriminant**

$$D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ or $f_{yy}(x_0, y_0) < 0$ there is a relative maximum at (x_0, y_0) ,
2. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ or $f_{yy}(x_0, y_0) > 0$ there is a relative minimum at (x_0, y_0) .

3. If $D < 0$ there is neither a maximum nor a minimum at (x_0, y_0) , but a saddle.
4. If $D = 0$, the test fails.

Note:

1. If the discriminant is positive at the point (x_0, y_0) and $f_{xx} < 0$, then the surface curves downward in all directions giving rise to a relative maximum.
2. If the discriminant is positive at the point (x_0, y_0) and $f_{xx} > 0$, then the surface curves upward in all directions giving rise to a relative minimum.
3. If the discriminant is negative at the point (x_0, y_0) , then the surface curves up in some directions and down in others. This behaviour creates a so-called *saddle point* at (x_0, y_0) .

Definition 11.4.5 Saddle Point.

Given the function $f(x, y)$, the point (x_0, y_0) is called a saddle point of f if

1. there is at least one point (x, y) arbitrarily close to (x_0, y_0) for which $f(x, y) > f(x_0, y_0)$, and
2. there is at least one point (x, y) arbitrarily close to (x_0, y_0) for which $f(x, y) < f(x_0, y_0)$.

Example 1. Verify that $f(x, y) = x^2 + y^2$ has a minimum at $(0, 0)$.

Solution. First, we compute all the needed derivatives:

$$f_x = 2x, f_y = 2y, f_{xx} = 2, f_{yy} = 2, f_{xy} = 0.$$

For critical point put $f_x = 0$ and $f_y = 0$ which gives the point $(0,0)$.

$$\text{Now } D(0,0) = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}^2(0,0) = 2 \cdot 2 - 0 = 4 > 0.$$

So, there is a relative minimum at $(0,0)$.

Example 2. Verify that $f(x, y) = x^2 - y^2$ neither has a maximum nor minimum at $(0,0)$.

Solution. First, we compute all the needed derivatives:

$$f_x = 2x, f_y = -2y, f_{xx} = 2, f_{yy} = -2, f_{xy} = 0.$$

For critical point put $f_x = 0$ and $f_y = 0$ which gives the point $(0,0)$.

$$\text{Now } D(0,0) = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}^2(0,0) = 2 \cdot (-2) - 0 = -4 < 0.$$

So, $(x, y) = x^2 - y^2$ neither has a maximum nor minimum at $(0,0)$ and $(0,0)$ is saddle point.

Example 3. Find the extrema of $f(x, y) = x^4 + y^4$.

Solution. First, we compute all the needed derivatives:

$$f_x = 4x^3, f_y = 4y^3, f_{xx} = 12x^2, f_{yy} = 12y^2, f_{xy} = 0.$$

For critical point put $f_x = 0$ and $f_y = 0$ which gives the point $(0,0)$.

$$\text{Now } D(0,0) = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}^2(0,0) = 0 \cdot 0 - 0 = 0.$$

So, we get no information. However, in this case it is easy to see that there is a minimum at $(0,0)$, because $f(0,0) = 0$ and at rest of the points $f(x, y) > 0$.

Example 4. A certain company produces and sells entertainment units. The total weekly revenue and cost in dollars is given by

$$R(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 280x + 260y + 1000,$$

and

$$C(x, y) = 160x + 160y + 6000$$

respectively, where x denotes the number of fully assembled units and y the number of kits produced and sold each week. Find the company's maximum profit per week, and the number of assembled units and kits that must be produced and sold to achieve this profit.

Solution. The weekly profit is given by

$$\begin{aligned} P(x, y) &= R(x, y) - C(x, y) \\ &= \left(-\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 280x + 260y + 1000\right) - \\ &\quad (160x + 160y + 6000) \\ &= -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 110y + 5000, \end{aligned}$$

where $x \geq 0$ and $y \geq 0$.

To find the relative maximum profit per week, we calculate the critical points by setting the first partial derivatives equal to zero:

$$P_x = -\frac{1}{2}x - \frac{1}{4}y + 120 = 0$$

$$P_y = -\frac{3}{4}y - \frac{1}{4}x + 100 = 0$$

Solving these equations, we get the only critical point (208,64).

We now apply the Second Derivative Test to confirm that the weekly maximum profit occurs at (208,64). For this we need the second partial derivatives of P :

$$P_{xx} = -\frac{1}{2}, P_{xy} = -\frac{1}{4}, P_{yy} = -\frac{3}{4}$$

Therefore, the discriminant is given by

$$D(x, y) = \left(-\frac{1}{2}\right)\left(-\frac{3}{4}\right) - \left(-\frac{1}{4}\right)^2 = \frac{3}{8} - \frac{1}{16} = \frac{5}{16} > 0$$

for all (x, y) in the domain of P and, in particular, $D(208, 64) > 0$. Since $P_{xx}(208, 64) = -12 < 0$, the point $(208, 64)$ leads to a relative maximum of P . Hence, the maximizing number of assembled units is 208 units and the maximizing number of kits is 64 kits, whose production and sale leads to a weekly profit of

$$\begin{aligned} P(208, 64) &= -\frac{1}{4}(208)^2 - \frac{3}{8}(64)^2 - \frac{1}{4}(208)(64) + 120(208) \\ &\quad + 110(64) - 5000 = \$10,680. \end{aligned}$$

11.5 CONSTRAINED OPTIMIZATION WITH LAGRANGE MULTIPLIERS

Differential calculus is also used to maximize or minimize a function subject to constraint. Given a function $f(x, y)$ subject to a constraint $g(x, y) = k$ (a constant), a new function F can be formed by

- (1) setting the constraint equal to zero,
- (2) multiplying it by λ (the Lagrange multiplier), and
- (3) adding the product to the original function:

$$F(x, y, \lambda) = f(x, y) + \lambda[k - g(x, y)] \quad (A)$$

Here $F(x, y, \lambda)$ is the Lagrangian function, $f(x, y)$ is the original or objective function, and $g(x, y)$ is the constraint. Since the constraint is

always set equal to zero, the product $\lambda[k - g(x, y)]$ also equals zero, and the addition of the term does not change the value of the objective function. Critical values x_0, y_0 , and λ_0 , at which the function is optimized, are found by taking the partial derivatives of F with respect to all three independent variables, setting them equal to zero, and solving simultaneously:

$$F_x(x, y, \lambda) = 0 \quad F_y(x, y, \lambda) = 0 \quad F_\lambda(x, y, \lambda) = 0$$

Example 1. Optimize the function

$$z = 4x^2 + 3xy + 6y^2$$

subject to the constraint $x + y = 56$.

- 1 Set the constraint equal to zero by subtracting the variables from the constant as in (A),

$$56 - x - y = 0$$

Multiply this difference by λ and add the product of the two to the objective function in order to form the Lagrangian function Z .

$$Z = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y) \quad (1)$$

- 2 Take the first-order partials, set them equal to zero, and solve simultaneously.

$$Z_x = 8x + 3y - \lambda = 0 \quad (2)$$

$$Z_y = 3x + 12y - \lambda = 0 \quad (3)$$

$$Z_\lambda = 56 - x - y = 0 \quad (4)$$

Subtracting (3) from (2) to eliminate λ gives

$$5x - 9y = 0 \quad x = 1.8y$$

Substitute $x = 1.8y$ in (4),

$$56 - 1.8y - y = 0 \quad y_0 = 20$$

From which we find

$$x_0 = 36 \quad \lambda_0 = 348$$

Substitute the critical values in (1),

$$\begin{aligned} Z &= 4(36)^2 + 3(36)(20) + 6(20)^2 + (348)(56 - 36 - 20) \\ &= 4(1296) + 3(720) + 6(400) + 348(0) = 9744 \end{aligned}$$

11.6 TOTAL AND PARTIALS DIFFERENTIALS AND TOTAL DERIVATIVE

In unit 8 we seen that the derivative dy/dx was presented as a single symbol denoting the limit of $\Delta y/\Delta x$ as Δx approaches zero. The derivative dy/dx may also be treated as a ratio of differentials in which dy is the differential of y and dx the differential of x . Given a function of a single independent variable $y = f(x)$, the differential of y , dy , measures the change in y resulting from a small change in x , written dx .

Given $y = 2x^2 + 5x + 4$, the differential of y is found by first taking the derivative of y with respect to x , which measures the rate at which y changes for a small change in x ,

$$\frac{dy}{dx} = 4x + 5 \quad \text{a derivative or rate of change}$$

and then multiplying that rate at which y changes for a small change in x by a specific change in $x(dx)$ to find the resulting change in $y(dy)$.

$$dy = (4x + 5)dx$$

Change in y = rate at which y changes for a small change in x . a small change in x .

Example 1. If $y = 4x^3 + 5x^2 - 7$, then $dy/dx = 12x^2 + 10x$ and the differential is

$$dy = (12x^2 + 10x)dx$$

Example 2. If $y = (2x - 5)^2$, then $dy/dx = 2(2x - 5)(2) = 8x - 20$ and the differential is

$$dy = (8x - 20)dx$$

Total and partial differentials

For a function of two or more independent variables, the total differential measures the change in the dependent variable brought about by a small change in each of the independent variables. If $z = f(x, y)$, the total differential dz is expressed mathematically as

$$dz = z_x dx + z_y dy$$

where z_x and z_y are the partial derivatives of z with respect to x and y respectively, and dx and dy are small changes in x and y . The total differential can thus be found by taking the partial derivatives of the function with respect to each independent variable and substituting these values in the formula above.

Example 1. Given: $z = x^4 + 8xy + 3y^3$

$$z_x = 4x^3 + 8y \quad z_y = 8x + 9y^2$$

which, when substituted in the total differential formula, gives

$$dz = (4x^3 + 8y)dx + (8x + 9y^2)dy$$

Example 2. Given: $z = (x - y)/(x + 1)$

The total differential is

$$z_x = \frac{(x+1)(1) - (x-y)(1)}{(x+1)^2} = \frac{y+1}{(x+1)^2}$$

$$z_y = \frac{(x+1)(-1) - (x-y)(0)}{(x+1)^2} = \frac{-1(x+1)}{(x+1)^2} = \frac{-1}{x+1}$$

$$dz = \frac{y+1}{(x+1)^2} dx - \left(\frac{1}{x+1} \right) dy$$

If one of the independent variables is held constant, for example, $dy = 0$, we then have a partial differential:

$$dz = z_x dx$$

A partial differential measures the change in the dependent variable of a multivariate function resulting from a small change in one of the independent variables and assumes the other independent variables are constant.

Given a case where $z = f(x, y)$ and $y = g(x)$, that is, when x and y are not independent, a change in x will affect z directly through the function f and indirectly through the function g . To measure the effect of a change in x on z when x and y are not independent, the total derivative must be found. The total derivative measures the direct effect of x on z , $\partial z / \partial x$, plus the indirect effect of x on z through y , $\frac{\partial z}{\partial y} \frac{dy}{dx}$. In brief, the total derivative is

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dx}$$

Example 3. An alternative method of finding the total derivative is to take the total differential of z

$$dz = z_x dx + z_y dy$$

and divide through mentally by dx . Thus,

$$\frac{dz}{dx} = z_x \frac{dx}{dx} + z_y \frac{dy}{dx}$$

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dx}$$

Since $dx/dx = 1$.

Example 4. Given

$$z = f(x, y) = 6x^3 + 7y$$

where $y = g(x) = 4x^2 + 3x + 8$, the total derivative dz/dx with respect to x is

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dx}$$

where $z_x = 18x^2$, $z_y = 7$, and $dy/dx = 8x + 3$. Substituting above,

$$\frac{dz}{dx} = 18x^2 + 7(8x + 3) = 18x^2 + 56x + 21$$

To check the answer, substitute $y = 4x^2 + 3x + 8$ in the original function to make z a function of x alone and then take the derivative as follows:

$$z = 6x^3 + 7(4x^2 + 3x + 8) = 6x^3 + 28x^2 + 21x + 56$$

Thus,

$$\frac{dz}{dx} = 18x^2 + 56x + 21.$$

Example 5. The total derivative can be expanded to accommodate other interconnections as well. Given

$$z = 8x^2 + 3y^2 \quad x = 4t \quad y = 5t$$

the total derivative of z with respect to t then becomes

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

where $z_x = 16x$, $z_y = 6y$, $dx/dt = 4$, and $dy/dt = 5$. Substituting above,

$$\frac{dz}{dt} = 16x(4) + 6y(5) = 64x + 30y$$

Then substituting $x = 4t$ and $y = 5t$ immediately above,

$$\frac{dz}{dt} = 64(4t) + 30(5t) = 406t.$$

Check Your Progress

1. For $z = x^2y + 3xy^2$, find $\partial z/\partial x$ and $\partial z/\partial y$.

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2. If $f(x, y) = x^3 + 2xy + y^2$, find f_x and f_y at $(1, 2)$.

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3. For $z = x^2y^3$, find the second-order partial derivatives $\partial^2 z/\partial x^2$ and $\partial^2 z/\partial y^2$.

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11.7 RULES FOR IMPLICIT AND INVERSE FUNCTION

As we know, functions of the form $y = f(x)$ express y explicitly in terms of x and are called explicit functions. Functions of the form $f(x, y) = 0$ do not express y explicitly in terms of x and are called implicit functions. If an implicit function $f(x, y) = 0$ exists and $f_y \neq 0$ at the point around which the implicit function is defined, the total differential is simply $f_x dx + f_y dy = 0$.

Recalling that a derivative is a ratio of differentials, we can then rearrange the terms to get the implicit function rule:

$$\frac{dy}{dx} = \frac{-f_x}{f_y} \quad (1)$$

Notice that the derivative dy/dx is the negative of the reciprocal of the corresponding partials.

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{1}{f_y/f_x}$$

Given a function $y = f(x)$, an inverse function $x = f^{-1}(y)$ exists if each value of y yields one and only one value of x . Assuming the inverse function exists, the inverse function rule states that the derivative of the inverse function is the reciprocal of the derivative of the original function. Thus, if $Q = f(P)$ is the original function, the derivative of the original function is dQ/dP , the derivative of the inverse function [$P = f^{-1}(Q)$] is dP/dQ , and

$$\frac{dP}{dQ} = \frac{1}{\frac{dQ}{dP}} \text{ provided } \frac{dQ}{dP} \neq 0 \quad (2)$$

Example 1. Given the implicit functions:

(a) $7x^2 - y = 0$

$$(b) 3x^4 - 7y^5 - 86 = 0$$

the derivative dy/dx is found as follows:

a) From (1),

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Here $f_x = 14x$ and $f_y = -1$. Substituting above,

$$\frac{dy}{dx} = -\frac{14x}{(-1)} = 14x$$

The function in this case was deliberately kept simple so the answer could easily be checked by solving for y in terms of x and then taking the derivative directly. Since $y = 7x^2$, $dy/dx = 14x$.

b)

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{12x^3}{-35y^4} = \frac{12x^3}{35y^4}.$$

Example 2. Find the derivative for the inverse of the following functions:

1. Given $Q = 20 - 2P$,

$$\frac{dP}{dQ} = \frac{1}{dQ/dP}$$

where $dQ/dP = -2$. Thus,

$$\frac{dP}{dQ} = \frac{1}{-2} = -\frac{1}{2}$$

2. Given $Q = 20 - 2P$,

$$\frac{dP}{dQ} = \frac{1}{dQ/dP} = \frac{1}{9P^2} \quad (P \neq 0)$$

11.8 SUMMARY

In this unit, we explored the essential concepts of multivariable functions and their applications in economics. Key topics included:

- **Partial derivatives:** Measuring the rate of change of a function with respect to one variable while holding others constant.
- **Optimization:** Finding maximum or minimum values of multivariable functions, crucial for-profit maximization and cost minimization.
- **Lagrange multipliers:** A method to optimize functions subject to constraints, commonly used in economic equilibrium analysis.
- **Total differentials and derivatives:** Tools to approximate changes in functions and link dependent variables.
- **Implicit and inverse functions:** Techniques to solve economic models where relationships between variables are not explicitly defined.

11.9 GLOSSARY

Multivariable Function: A function depending on two or more independent variables.

Partial Derivative: The derivative with respect to one variable while other variables are held constant.

Total Differential: An expression representing the combined change in a function due to changes in its variables.

First Partial Derivative: A first-order rate of change with respect to one variable.

Second Partial Derivative: A derivative obtained by differentiating a first partial derivative.

Mixed Partial Derivative: A derivative involving differentiation with respect to two different variables.

Gradient: A vector of first partial derivatives of a scalar function.

Chain Rule: A rule used to differentiate composite multivariable functions.

11.10 EXERCISES

1. Calculate the partial derivatives of the following function:

$$f(x, y) = 3x^2y + 4xy^2 - 5y.$$

2. Classify the critical points of the following function:

$$f(x, y) = x^2 + 4y^2 - 2x + 8y - 1$$

3. Classify the critical points of the following function:

$$f(x, y) = x^2 - y^2 + 6x - 10y + 2$$

4. Classify the critical points of the following function:

$$f(x, y) = 9 + 4x - y - 2x^2 - 3y^2$$

5. Find the absolute maximum and minimum points of $f = x^2 + 3y - 3xy$ over

the region bounded by $y = x$, $y = 0$, and $x = 2$.

6. A certain company produces and sells flat screen TVs. The total weekly

revenue and cost in dollars is given by

$$R(x, y) = -0.2x^2 - 0.25y^2 - 0.2xy + 225x + 135y +$$

1500, and

$$C(x, y) = 125x + 45y + 5500$$

respectively, where x denotes the number of deluxe units and y the number of standard units produced and sold each week. Find the company's maximum profit per week, and the

number of deluxe and standard units that must be produced and sold to achieve this profit.

7. Optimize the function $f(x, y) = xy$ subject to the constraint $x + y = 10$.
8. Differentiate implicitly: $x^2 + y^2 = 25$.

11.11 SUGGESTED READINGS

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2. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007.
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UNIT 12 APPLICATIONS OF MULTIVARIABLE CALCULUS IN ECONOMICS

STRUCTURE

- 12.1. Introduction
- 12.2. Objectives
- 12.3. Marginal productivity
- 12.4. Income determination multipliers and comparative statics
- 12.5. Income and cross price elasticities of demand
- 12.6. Differentials and incremental changes
- 12.7. Optimization of multivariable functions in economics
- 12.8. Constrained optimization of multivariable functions in economics
- 12.9. Homogeneous production functions
- 12.10. Summary
- 12.11. Glossary
- 12.12. Exercises
- 12.13. Suggested Readings

12.1 INTRODUCTION

Multivariable calculus extends the principles of single-variable calculus to functions involving multiple variables. This branch of calculus is essential for analyzing complex economic models where multiple factors interact simultaneously. In economics, multivariable calculus helps measure marginal productivity, compute elasticities, analyze income determination, and optimize functions with or without constraints. This unit explores these applications, providing tools to evaluate how small changes in inputs influence economic outputs, optimize production, and maximize profit. Understanding constrained and unconstrained optimization is vital for resource allocation and decision-making in economic systems.

12.2 OBJECTIVES

After completing this unit, you will be able to

- Calculate marginal productivity for functions involving multiple variables.
- Apply income determination multipliers and comparative statics to economic models.
- Compute income and cross-price elasticities of demand.
- Use differentials to estimate incremental changes in economic functions.
- Optimize economic functions involving multiple variables.
- Perform constrained optimization using methods like Lagrange multipliers.
- Analyze and apply homogeneous production functions in economic contexts.

12.3 MARGINAL PRODUCTIVITY

The marginal product of capital (MP_K) is defined as the change in output brought about by a small change in capital when all the other factors of production are held constant. Given a production function such as

$$Q = 36KL - 2K^2 - 3L^2$$

the MP_K is measured by taking the partial derivative $\partial Q / \partial K$. Thus,

$$MP_K = \frac{\partial Q}{\partial K} = 36L - 4K$$

Similarly, for labor, $MP_L = \partial Q / \partial L = 36K - 6L$.

12.4 INCOME DETERMINATION MULTIPLIERS AND COMPARATIVE STATICS

The partial derivative can also be used to derive the various multipliers of an income determination model. In calculating how the equilibrium level of the endogenous variable can be expected to change in response to a change in any of the exogenous variables or parameters, income determination multipliers provide an elementary Exercises in what is called comparative static analysis or, more simply, comparative statics. Given

$$Y = C + I + G + (X - Z)$$

where

$$\begin{aligned} C &= C_0 + bY & G &= G_0 & Z &= Z_0 \\ I &= I_0 + aY & X &= X_0 \end{aligned}$$

Using simple substitution as in Problem 2.19, the equilibrium level of income is

$$\bar{Y} = \frac{1}{1 - b - a} (C_0 + I_0 + G_0 + X_0 - Z_0) \quad (1)$$

Taking the partial derivative of (6.1) with respect to any of the variables or parameters gives the multiplier for that variable or parameter. Thus, the government multiplier is given by

$$\frac{\partial \bar{Y}}{\partial G_0} = \frac{1}{1 - b - a}$$

The import multiplier is given by

$$\frac{\partial \bar{Y}}{\partial Z_0} = -\frac{1}{1 - b - a}$$

And the multiplier for a change in the marginal propensity to invest is given by $\partial \bar{Y} / \partial a$, where, by means of the quotient rule,

$$\begin{aligned}\frac{\partial \bar{Y}}{\partial a} &= \frac{(1 - b - a)(0) - (C_0 + I_0 + G_0 + X_0 - Z_0)(-1)}{(1 - b - a)^2} \\ &= \frac{C_0 + I_0 + G_0 + X_0 - Z_0}{(1 - b - a)^2}\end{aligned}$$

This can alternately be expressed as

$$\frac{\partial \bar{Y}}{\partial a} = \frac{1}{1 - b - a} (C_0 + I_0 + G_0 + X_0 - Z_0) \left(\frac{1}{1 - b - a} \right)$$

which from (1) reduces to

$$\frac{\partial \bar{Y}}{\partial a} = \frac{\bar{Y}}{1 - b - a}$$

Example: Given a simple model

$$\begin{aligned}Y &= C + I_0 + G_0 & Yd &= Y - T \\ C &= C_0 + bYd & T &= T_0\end{aligned}$$

where taxation does not depend on income, calculate the effect on the equilibrium level of income of a 1-unit change in government expenditure exactly offset by a 1-unit change in taxation. That is, do the comparative-static analysis of finding the balanced-budget multiplier for an economy in which there is only autonomous taxation.

$$\begin{aligned}Y &= C_0 + b(Y - T_0) + I_0 + G_0 \\ \bar{Y} &= \frac{1}{1 - b} (C_0 - bT_0 + I_0 + G_0)\end{aligned}$$

Thus, the government multiplier is

$$\frac{\partial \bar{Y}}{\partial G_0} = \frac{1}{1 - b}$$

and the tax multiplier is

$$\frac{\partial \bar{Y}}{\partial T_0} = \frac{-b}{1-b}$$

The balanced-budget effect of a 1-unit increase in government spending matched by a 1-unit increase in taxation is the sum of (6.24) and (6.25). Therefore,

$$\Delta \hat{Y} = \frac{1}{1-b} + \left(\frac{-b}{1-b} \right) = \frac{1}{1-b} - \frac{b}{1-b} = \frac{1-b}{1-b} = 1$$

A change in government expenditure matched by an equal change in government taxation will have a positive effect on the equilibrium level of income exactly equal to the change in government expenditure and taxation. The multiplier in this case is +1.

12.5 INCOME AND CROSS PRICE ELASTICITIES OF DEMAND

Income elasticity of demand ϵ_Y measures the percentage change in the demand for a good resulting from a small percentage change in income, when all other variables are held constant. Cross price elasticity of demand ϵ_c measures the relative responsiveness of the demand for one product to changes in the price of another, when all other variables are held constant. Given the demand function

$$Q_1 = a - bP_1 + cP_2 + mY$$

where Y = income and P_2 = the price of a substitute good, the income elasticity of demand is

$$\epsilon_Y = \frac{\partial Q_1}{Q_1} \div \frac{\partial Y}{Y} = \frac{\partial Q_1}{\partial Y} \left(\frac{Y}{Q_1} \right)$$

and the cross price elasticity of demand is

$$\epsilon_c = \frac{\partial Q_1}{Q_1} \div \frac{\partial P_2}{P_2} = \frac{\partial Q_1}{\partial P_2} \left(\frac{P_2}{Q_1} \right)$$

Example 1. Given the demand for beef

$$Q_b = 4850 - 5P_b + 1.5P_p + 0.1Y \quad (1)$$

with $Y = 10,000$, $P_b = 200$, and the price of pork $P_p = 100$. The calculations for (1) the income elasticity and (2) the cross-price elasticity of demand for beef are given below.

1)

$$\epsilon_Y = \frac{\partial Q_b}{Q_b} \div \frac{\partial Y}{Y} = \frac{\partial Q_b}{\partial Y} \left(\frac{Y}{Q_b} \right) \quad (2)$$

From (1),

$$\frac{\partial Q_b}{\partial Y} = 0.1$$

and

$$Q_b = 4850 - 5(200) + 1.5(100) + 0.1(10,000) = 5000 \quad (3)$$

Substituting in (6.3), $\epsilon_Y = 0.1(10,000/5000) = 0.2$.

With $\epsilon_Y < 1$, the good is income-inelastic. For any given percentage increase in national income, demand for the good will increase less than proportionately. Hence the relative market share of the good will decline as the economy expands. Since the income elasticity of demand suggests the growth potential of a market, the growth potential in this case is limited.

2)

$$\epsilon_c = \frac{\partial Q_b}{Q_b} \div \frac{\partial P_p}{P_p} = \frac{\partial Q_b}{\partial P_p} \left(\frac{P_p}{Q_b} \right)$$

From (1), $\partial Q_b / \partial P_p = 1.5$; from (3), $Q_b = 5000$. Thus,

$$\epsilon_c = 1.5 \left(\frac{100}{5000} \right) = 0.03$$

For substitute goods, such as beef and pork, $\partial Q_1/\partial P_2 > 0$ and the cross-price elasticity will be positive. For complementary goods, $\partial Q_1/\partial P_2 < 0$ and the cross-price elasticity will be negative. If $\partial Q_1/\partial P_2 = 0$, the goods are unrelated.

Example 2. Continuing with Example 1, the percentage change in the demand for beef resulting from a 10 percent increase in the price of pork is estimated as follows:

$$\epsilon_c = \frac{\partial Q_b}{Q_b} \div \frac{\partial P_p}{P_p}$$

Rearranging terms and substituting the known parameters,

$$\frac{\partial Q_b}{Q_b} = \epsilon_c \frac{\partial P_p}{P_p} = (0.03)(0.10) = 0.003$$

The percentage change in the demand for beef $\partial Q_b/Q_b$ will be 0.3 percent.

12.6 DIFFERENTIALS AND INCREMENTAL CHANGES

Frequently in economics we want to measure the effect on the dependent variable (costs, revenue, profit) of a change in an independent variable (labor hired, capital used, items sold). If the change is a relatively small one, the differential will measure the effect. Thus, if $z = f(x, y)$, the effect on z of a small change in x is given by the partial differential

$$dz = z_x dx$$

The effect of larger changes can be approximated by multiplying the partial derivative by the proposed change. Thus,

$$\Delta z \approx z_x \Delta x$$

If the original function $z = f(x, y)$ is linear,

$$\frac{dz}{dx} = \frac{\Delta z}{\Delta x}$$

and the effect of the change will be measured exactly:

$$\Delta z = z_x \Delta x.$$

Example 1. A firm's costs are related to its output of two goods x and y . The functional relationship is

$$TC = x^2 - 0.5xy + y^2$$

The additional cost of a slight increment in output x will be given by the differential

$$dTC = (2x - 0.5y)dx$$

The costs of larger increments can be approximated by multiplying the partial derivative with respect to x by the change in x . Mathematically,

$$\Delta TC \approx \frac{\partial TC}{\partial x} \Delta x \quad (1)$$

Since $\partial TC / \partial x =$ the marginal cost (MC_x) of x , we can also write (1) as

$$\Delta TC \approx MC_x \Delta x$$

If initially $x = 100$, $y = 60$, and $\Delta x = 3$, then

$$\Delta TC \approx [2(100) - 0.5(60)] \cong 510$$

Example 2. Assume in Section 6.2 that $b = 0.7$, $a = 0.1$, and $Y = 1200$. The differential can then be used to calculate the effect of an increase in any of the independent variables. Given the partial derivative

$$\frac{\partial \bar{Y}}{\partial G_0} = \frac{1}{1 - b - a}$$

the partial differential is

$$d\bar{Y} = \frac{1}{1 - b - a} dG_0$$

In a linear model such as this, where the slope is everywhere constant,

$$\frac{\partial \bar{Y}}{\partial G_0} = \frac{\Delta \bar{Y}}{\Delta G_0}$$

Hence

$$\Delta \bar{Y} = \frac{1}{1 - b - a} \Delta G_0$$

If the government increases expenditures by \$100,

$$\Delta \bar{Y} = \frac{1}{1 - 0.7 - 0.1} (100) = 500$$

12.7 OPTIMIZATION OF MULTIVARIABLE FUNCTIONS IN ECONOMICS

Food processors frequently sell different grades of the same product: quality, standard, economy; some, too, sell part of their output under their own brand name and part under the brand name of a large chain store. Clothing manufacturers and designers frequently have a top brand and cheaper imitations for discount department stores. Maximizing profits or minimizing costs under these conditions involve functions of more than one variable. Thus, the basic rules for optimization of multivariate functions are required.

Example 1. A firm producing two goods x and y has the profit function

$$\pi = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$$

To find the profit-maximizing level of output for each of the two goods and test to be sure profits are maximized:

1. Take the first-order partial derivatives, set them equal to zero, and solve for x and y simultaneously.

$$\pi_x = 64 - 4x + 4y = 0 \quad (1)$$

$$\pi_y = 4x - 8y + 32 = 0 \quad (2)$$

When solved simultaneously, $\bar{x} = 40$ and $\bar{y} = 24$.

2. Take the second-order direct partial derivatives and make sure both are negative, as is required for a relative maximum. From (1) and (2),

$$\pi_{xx} = -4 \quad \pi_{yy} = -8$$

3. Take the cross partials to make sure $\pi_{xx}\pi_{yy} > (\pi_{xy})^2$. From (6.6) and (6.7), $\pi_{xy} = 4 = \pi_{yx}$. Thus,

$$\begin{aligned} \pi_{xx}\pi_{yy} &> (\pi_{xy})^2 \\ (-4)(-8) &> (4)^2 \\ 32 &> 16 \end{aligned}$$

Profits are indeed maximized at $\bar{x} = 40$ and $\bar{y} = 24$. At that point, $\pi = 1650$.

Example 2. In monopolistic competition producers must determine the price that will maximize their profit. Assume that a producer offers two different brands of a product, for which the demand functions are

$$Q_1 = 14 - 0.25P_1 \quad (1)$$

$$Q_2 = 24 - 0.5P_2 \quad (2)$$

and the joint cost function is

$$TC = Q_1^2 + 5Q_1Q_2 + Q_2^2 \quad (3)$$

The profit-maximizing level of output, the price that should be charged for each brand, and the profits are determined as follows:

First, establish the profit function π in terms of Q_1 and Q_2 . Since π = total revenue (TR) minus total cost (TC) and the total revenue for the firm is $P_1Q_1 + P_2Q_2$, the firm's profit is

$$\pi = P_1Q_1 + P_2Q_2 - TC$$

Substituting from (6.10),

$$\pi = P_1 Q_1 + P_2 Q_2 - (Q_1^2 + 5Q_1 Q_2 + Q_2^2) \quad (4)$$

Next find the inverse functions of (1) and (2) by solving for P in terms of Q . Thus, from (1),

$$P_1 = 56 - 4Q_1 \quad (5)$$

$$P_2 = 48 - 2Q_2 \quad (6)$$

and from (3),

Substituting in (4),

$$\begin{aligned} \pi &= (56 - 4Q_1)Q_1 + (48 - 2Q_2)Q_2 - Q_1^2 - 5Q_1 Q_2 - Q_2^2 \\ &= 56Q_1 - 5Q_1^2 + 48Q_2 - 3Q_2^2 - 5Q_1 Q_2 \end{aligned} \quad (7)$$

Then maximize (7) by the familiar rules:

$$\pi_1 = 56 - 10Q_1 - 5Q_2 = 0 \quad \pi_2 = 48 - 6Q_2 - 5Q_1 = 0$$

which, when solved simultaneously, give $\bar{Q}_1 = 2.75$ and $\bar{Q}_2 = 5.7$.

Take the second derivatives to be sure π is maximized:

$$\pi_{11} = -10 \quad \pi_{22} = -6 \quad \pi_{12} = -5 = \pi_{21}$$

With both second direct partials negative and $\pi_{11}\pi_{22} > (\pi_{12})^2$, the function is maximized at the critical values.

Finally, substitute $\bar{Q}_1 = 2.75$ and $\bar{Q}_2 = 5.7$ in (6.12) and (6.13), respectively, to find the profit-maximizing price.

$$P_1 = 56 - 4(2.75) = 45 \quad P_2 = 48 - 2(5.7) = 36.6$$

Prices should be set at \$45 for brand 1 and \$36.60 for brand 2, leading to sales of 2.75 of brand 1 and 5.7 of brand 2. From (4) or (7), the maximum profit is

$$\pi = 45(2.75) + 36.6(5.7) - (2.75)^2 - 5(2.75)(5.7) - (5.7)^2 = 213.9$$

Check Your Progress

1. For the production function $Q = 10K^{0.5}L^{0.5}$, find the marginal products $\partial Q/\partial K$ and $\partial Q/\partial L$.

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2. If $C(x, y) = 100 + 4x^2 + 2y^2 + xy$, find the marginal cost with respect to x and y at $(2, 3)$.

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3. For $\pi(x, y) = 40x + 30y - x^2 - y^2$, find the stationary point by setting the first-order partial derivatives equal to zero.

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12.8 CONSTRAINED OPTIMIZATION OF MULTIVARIABLE FUNCTIONS IN ECONOMICS

Solutions to economic problems frequently have to be found under constraints (e.g., maximizing utility subject to a budget constraint or minimizing costs subject to some such minimal requirement of output as a production quota). Use of the Lagrangian function (see Unit 10-11) greatly facilitates this task.

Example 1. Find the critical values for minimizing the costs of a firm producing two goods x and y when the total cost function is $c = 8x^2 - xy + 12y^2$ and the firm is bound by contract to produce a minimum combination of goods totaling 42, that is, subject to the constraint $x + y = 42$.

Set the constraint equal to zero, multiply it by λ , and form the Lagrangian function,

$$C = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

Take the first-order partials,

$$C_x = 16x - y - \lambda = 0$$

$$C_y = -x + 24y - \lambda = 0$$

$$C_\lambda = 42 - x - y = 0$$

Solving simultaneously, $\bar{x} = 25$, $\bar{y} = 17$, and $\bar{\lambda} = 383$. With $\bar{\lambda} = 383$, a 1-unit increase in the constraint or production quota will lead to an increase in cost of approximately \$383.

12.9 HOMOGENEOUS PRODUCTION FUNCTIONS

A production function is said to be homogeneous if when each input factor is multiplied by a positive real constant k , the constant can be completely factored out. If the exponent of the factor is 1, the function is homogeneous of degree 1; if the exponent of the factor is greater than 1, the function is homogeneous of degree greater than 1; and if the exponent of the factor is less than 1, the function is homogeneous of degree less than 1. Mathematically, a function $z = f(x, y)$ is homogeneous of degree n if for all positive real values of k , $f(kx, ky) = k^n f(x, y)$.

Example. The degree of homogeneity of a function is illustrated below.

1. $z = 8x + 9y$ is homogeneous of degree 1 because

$$f(kx, ky) = 8kx + 9ky = k(8x + 9y)$$

2. $z = x^2 + xy + y^2$ is homogeneous of degree 2 because

$$f(kx, ky) = (kx)^2 + (kx)(ky) + (ky)^2 = k^2(x^2 + xy + y^2)$$

3. $z = x^{0.3}y^{0.4}$ is homogeneous of degree less than 1 because

$$f(kx, ky) = (kx)^{0.3}(ky)^{0.4} = k^{0.3+0.4}(x^{0.3}y^{0.4}) = k^{0.7}(x^{0.3}y^{0.4})$$

4. $z = 2x/y$ is homogeneous of degree 0 because

$$f(kx, ky) = \frac{2kx}{ky} = 1 \left(\frac{2x}{y} \right) \text{ since } \frac{k}{k} = k^0 = 1$$

5. $z = x^3 + 2xy + y^3$ is not homogeneous because k cannot be completely factored out:

$$\begin{aligned} f(kx, ky) &= (kx)^3 + 2(kx)(ky) + (ky)^3 \\ &= k^3x^3 + 2k^2xy + k^3y^3 = k^2(kx^3 + 2xy + ky^3) \end{aligned}$$

6. $Q = AK^\alpha L^\beta$ is homogeneous of degree $\alpha + \beta$ because

$$Q(kK, kL) = A(kK)^\alpha (kL)^\beta = Ak^\alpha K^\alpha k^\beta L^\beta = k^{\alpha+\beta} (AK^\alpha L^\beta)$$

12.10 SUMMARY

In this unit, we have learned about

- **Marginal Productivity:** Measures the additional output resulting from a small increase in one input while holding others constant.
- **Income Determination and Multipliers:** Analyzes how changes in economic inputs influence national income through multipliers and comparative statics.
- **Elasticities of Demand:**
Income Elasticity: Measures the responsiveness of demand to changes in income.

Cross-Price Elasticity: Evaluates how the demand for one good responds to changes in the price of another.

- **Differentials and Incremental Changes:** Small changes in inputs lead to proportional changes in outputs, approximated by differentials.
- **Optimization of Multivariable Functions:** Identifies maxima and minima of functions with several variables to maximize profit or minimize cost.
- **Constrained Optimization:** Applies Lagrange multipliers to optimize economic functions subject to constraints, such as budget limits.
- **Homogeneous Production Functions:** Functions that exhibit constant returns to scale, allowing for proportional changes in output when all inputs change by the same factor.

12.11 GLOSSARY

Production Function: A function relating output to quantities of inputs.

Marginal Product: The additional output from an additional unit of an input.

Partial Elasticity: Responsiveness of an economic variable to a change in one variable, holding others constant.

Marginal Utility: The additional utility obtained from one more unit of a commodity.

Constrained Optimization: Optimization subject to specified restrictions.

Lagrange Multiplier: A method for solving constrained optimization problems.

Isoquant: A curve showing input combinations producing the same output.

Optimal Point: A point at which an objective function attains a desired maximum or minimum.

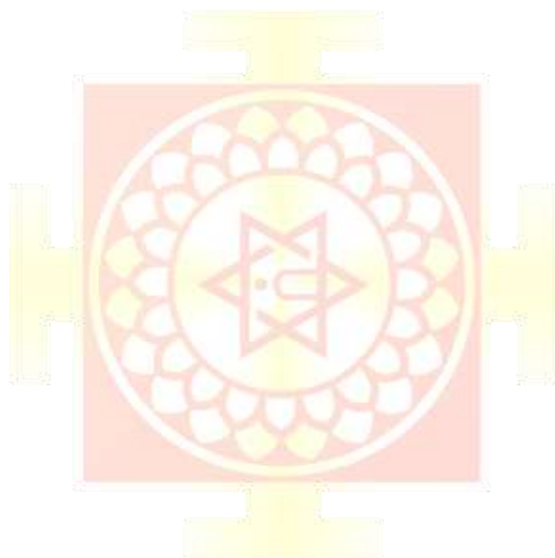
12.12 EXERCISES

- 1 Calculate marginal productivity for the following production functions:
 - a) $Q(L, K) = 4L^{0.5}K^{0.5}$
 - b) $Q(L, K) = 2L^2 + 3K^2$
- 2 Apply comparative statics to analyze the effects of changes in income on the following functions:
 - a) $Y = C + I + G$ where $C = 100 + 0.8Y$
 - b) $Q = 500 - 2P + 0.5I$, find $\frac{dQ}{dI}$.
- 3 Compute the income and cross-price elasticities for the demand functions:
 - a) $Q_d = 300 - 2P + 0.6I$
 - b) $Q_x = 200 - 3P_x + 2P_y$
- 4 Use differentials to estimate changes in output for:
 - a) $Q(L, K) = L^{0.5}K^{0.5}$ at $L = 100, K = 25$ with $dL = 1, dK = 0.5$.
 - b) $Q(L, K) = L^2 + K^2$ at $L = 50, K = 20$ with $dL = 2, dK = 1$.
- 5 Optimize multivariable functions to find maximum output:
 - a) $Q(L, K) = 2L^{0.5}K^{0.5}$
 - b) $P(x, y) = x^2 + y^2 - 4x - 6y$
- 6 Perform constrained optimization using Lagrange multipliers for:
 - a) Maximize $Q(L, K) = L^{0.5}K^{0.5}$ subject to $2L + K = 100$.
 - b) Maximize $P(x, y) = x^2 + 3y^2$ subject to $x + y = 10$.

12.13 SUGGESTED READINGS

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BLOCK-IV INTEGRATION

UNIT 13 INTEGRATION AND BASIC FORMULAE

STRUCTURE

- 13.1. Introduction
 - 13.2. Objectives
 - 13.3. Integration: Indefinite Integral
 - 13.4. Summary
 - 13.5. Glossary
 - 13.6. Exercises
 - 13.7. Suggested Readings
-

13.1 INTRODUCTION

Integration and differentiation are two fundamental operations in calculus. They are inverse operations of each other in a certain sense.

Differentiation is the process of finding the derivative of a function, which measures how the function's output changes with respect to its input. The derivative of a function $f(x)$ is denoted by $f'(x)$ or $\frac{dy}{dx}$, where y is the dependent variable and x is the independent variable.

Integration, on the other hand, is the process of finding the antiderivative of a function. The antiderivative $F(x)$ of a function $f(x)$ is a function whose derivative is equal to $f(x)$. This process is also known as finding the indefinite integral of a function. The symbol $\int$ represents the process of integration, and $F(x) = \int f(x) dx$ denotes the antiderivative of $f(x)$.

13.2 OBJECTIVES

After completing this unit, students will be able to

- Understand the concept of integration
- Remember basic integration formulae
- Apply integration formulae to solve numerical problems

13.3 INTEGRATION: INDEFINITE INTEGRAL

If the function $f(x)$ be the derivative of the function $g(x)$ w.r.t. x on a certain interval of the x -axis, then we say that an integral of $f(x)$ w.r.t. x is $g(x)$ and write it symbolically as

$$\int f(x) dx = g(x).$$

The variable x is called the variable of integration. It is customary to use the differential ' dx ' of the variable ' x ' in the process of integration. The differential ' dx ' indicate that the integration is to be carried w.r.t. x . The symbol $\int$ denote the process of integration. The function whose integral is to be found out is called the integrand. If the integral of $f(x)$ w.r.t. x is $g(x)$, then we also say that $g(x)$ is an antiderivative of $f(x)$.

Examples (i) x^3 is the derivative of the function $\frac{x^4}{4}$ w.r.t. x .

Therefore, $\frac{x^4}{4}$ is an integral of x^3 w.r.t. x i.e.,

$$\int x^3 dx = \frac{x^4}{4}.$$

(ii) Since $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x dx = \sin x$, i.e. $\sin x$ is an integral of $\cos x$.

Since $\frac{d}{dx}(\sin x) = \cos x$, we have $\int \cos x dx = \sin x$.

Moreover, if C is any constant then $\frac{d}{dx}(\sin x + C) = \cos x$.

So, in general, $\int \cos x dx = (\sin x + C)$.

Clearly, different values of C will give different integrals.

Thus, a given function may have an indefinite number of integrals.

Because of this property, we call these integrals indefinite integrals.

Thus, $\frac{d}{dx}[g(x)] = f(x) \Rightarrow \int f(x)dx = g(x) + C$, where C is a constant, called the constant of integration. Any function to be integrated is known as an integrand.

It is very important to note that the integral of a function may or may not exist. In other words, it may not be possible to find a function whose derivative is equal to the given function.

On the basis of differentiation and the definition of integration, we have the following results.

$$1 \quad \frac{d}{dx}(C) = 0 \Rightarrow \int 0 dx = C.$$

$$2 \quad \frac{d}{dx}(x) = 1 \Rightarrow \int 1 dx = x + C.$$

$$3 \quad \frac{d}{dx}(kx) = k \Rightarrow \int k dx = kx + C.$$

$$4 \quad \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1 \Rightarrow \int x^n dx = \frac{x^{n+1}}{(n+1)} + C.$$

$$5 \quad \frac{d}{dx}(\log|x|) = \frac{1}{x} \Rightarrow \int \frac{1}{x} dx = \log|x| + C.$$

$$6 \quad \frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + C.$$

$$7 \quad \frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x \Rightarrow \int a^x dx = \frac{a^x}{\log a} + C.$$

$$8 \quad \frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x dx = \sin x + C$$

$$9 \quad \frac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x dx = -\cos x + C$$

$$10 \quad \frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x dx = \tan x + C$$

$$11 \quad \frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x \Rightarrow \int \operatorname{cosec}^2 x dx = -\cot x + C$$

$$12 \quad \frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x dx = \sec x + C$$

$$13 \quad \frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x \Rightarrow \int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$$

$$14 \quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$15 \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{(1+x^2)} \Rightarrow \int \frac{1}{(1+x^2)} dx = \tan^{-1} x + C$$

$$16 \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \Rightarrow \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$$

Some Standard Results on Integration

1. $\frac{d}{dx} \left\{ \int f(x) dx \right\} = f(x).$
2. $\int k \cdot f(x) dx = k \cdot \int f(x) dx$, where k is a constant.
 $\int \{f_1(x) + f_2(x)\} dx = \int f_1(x) dx + \int f_2(x) dx.$
3. $\int \{f_1(x) - f_2(x)\} dx = \int f_1(x) dx - \int f_2(x) dx.$
4. $\int \{k_1 \cdot f_1(x) \pm k_2 \cdot f_2(x) \pm \dots \pm k_n \cdot f_n(x)\} dx = k_1 \cdot \int f_1(x) dx \pm k_2 \cdot \int f_2(x) dx \pm \dots \pm k_n \cdot \int f_n(x) dx.$

Example 1: $\int x^3 dx = \frac{x^{(3+1)}}{(3+1)} + C = \frac{x^4}{4} + C.$

Example 2: $\int \sqrt[3]{x} dx = \int x^{1/3} dx = \frac{x^{(\frac{1}{3}+1)}}{(\frac{1}{3}+1)} + C = \frac{3}{4} x^{4/3} + C.$

Example 3: $\int dx = \int x^0 dx = \frac{x^{(0+1)}}{(0+1)} + C = x + C.$

Example 4: $\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{(-3+1)}}{(-3+1)} + C = -\frac{1}{2x^2} + C.$

Example 5: $\int \frac{1}{x^{1/2}} dx = \int x^{-1/2} dx = \frac{x^{(-\frac{1}{2}+1)}}{(-\frac{1}{2}+1)} + C = 2x^{1/2} + C.$

Example 6: $\int 5^x dx = \frac{5^x}{\log 5} + C.$

Example 7: $\int \left(5x^3 + 2x^{-5} - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x} \right) dx$

$$= 5 \int x^3 dx + 2 \int x^{-5} dx - 7 \int x dx + \int x^{-1/2} dx + 5 \int \frac{1}{x} dx$$

$$= 5 \cdot \frac{x^4}{4} + 2 \cdot \frac{x^{-4}}{(-4)} - 7 \cdot \frac{x^2}{2} + \frac{x^{1/2}}{(1/2)} + 5 \log |x| + C$$

$$= \frac{5x^4}{4} - \frac{1}{2x^4} - \frac{7x^2}{2} + 2\sqrt{x} + 5 \log |x| + C.$$

Example 8:

$$\begin{aligned} \int (1-x)(2+3x)(5-4x) dx &= \int (10-3x-19x^2+12x^3) dx \\ &= 10 \int dx - 3 \int x dx - 19 \int x^2 dx + 12 \int x^3 dx \\ &= 10x - 3 \cdot \frac{x^2}{2} - 19 \cdot \frac{x^3}{3} + 12 \cdot \frac{x^4}{4} + C \\ &= 10x - \frac{3x^2}{2} - \frac{19x^3}{3} + 3x^4 + C. \end{aligned}$$

Example 9: $\int \left(\frac{3x^4 - 5x^3 + 4x^2 - x + 2}{x^3} \right) dx = \int \left(3x - 5 + \frac{4}{x} - \frac{1}{x^2} + \frac{2}{x^3} \right) dx$

[dividing each term by x^3]

$$\begin{aligned} &= 3 \int x dx - 5 \int dx + 4 \int \frac{1}{x} dx - \int x^{-2} dx + 2 \int x^{-3} dx \\ &= 3 \cdot \frac{x^2}{2} - 5x + 4 \log |x| - \left(-\frac{1}{x} \right) + 2 \left(\frac{x^{-2}}{-2} \right) + C \\ &= \frac{3x^2}{2} - 5x + 4 \log |x| + \frac{1}{x} - \frac{1}{x^2} + C. \end{aligned}$$

Example 10: $\int \left(\frac{x^4+1}{x^2+1} \right) dx = \int \left[x^2 - 1 + \frac{2}{(x^2+1)} \right] dx$

$$= \int x^2 dx - \int dx + 2 \int \frac{1}{x^2 + 1} dx$$

$$= \frac{x^3}{3} - x + 2 \tan^{-1} x + C.$$

Example 11: $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \int \sec^2 x dx - \int dx = \tan x - x + C.$

Example 12: $\int \cot^2 x dx = \int (\operatorname{cosec}^2 x - 1) dx = \int \operatorname{cosec}^2 x dx - \int dx = -\cot x - x + C.$

Example 13: Evaluate $\int \sqrt{1 - \sin 2x} dx.$

Solution:

$$\begin{aligned} \int \sqrt{1 - \sin 2x} dx &= \int (\cos^2 x + \sin^2 x - 2 \sin x \cos x)^{1/2} dx \\ &= \int \sqrt{(\cos x - \sin x)^2} dx \\ &= \int (\cos x - \sin x) dx = \int \cos x dx - \int \sin x dx \\ &= \sin x - (-\cos x) + C = \sin x + \cos x + C. \end{aligned}$$

Example 14: Evaluate $\int \frac{dx}{1 + \sin x}$

Solution:

$$\begin{aligned} \int \frac{dx}{1 + \sin x} &= \int \frac{1}{(1 + \sin x)} \times \frac{(1 - \sin x)}{(1 - \sin x)} dx \\ &= \int \frac{(1 - \sin x)}{(1 - \sin^2 x)} dx = \int \frac{(1 - \sin x)}{\cos^2 x} dx \\ &= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx = \int (\sec^2 x - \sec x \tan x) dx \\ &= \int \sec^2 x dx - \int \sec x \tan x dx = \tan x - \sec x + C. \end{aligned}$$

Check Your Progress

1. Evaluate: $\int (3x^2 + 4x - 5) dx$.

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2. Evaluate: $\int (1/x + e^x) dx$.

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3. If $MC(q) = 4q + 10$ and $C(0) = 50$, find the total cost function $C(q)$ by integration.

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.....

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Example 15: Evaluate $\int \frac{\sec x}{(\sec x + \tan x)} dx$.

Solution:

$$\begin{aligned} \int \frac{\sec x}{(\sec x + \tan x)} dx &= \int \frac{\sec x}{(\sec x + \tan x)} \times \frac{(\sec x - \tan x)}{(\sec x - \tan x)} dx \\ &= \int \frac{(\sec^2 x - \sec x \tan x)}{(\sec^2 x - \tan^2 x)} dx \end{aligned}$$

$$\begin{aligned}
 &= \int (\sec^2 x - \sec x \tan x) dx \\
 &= \int \sec^2 x dx - \int \sec x \tan x dx = \tan x - \sec x + C.
 \end{aligned}$$

Example 16: Evaluate $\int \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right) dx$

Solution:

$$\begin{aligned}
 \int \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right) dx &= \int \frac{2\sin^2 x}{2\cos^2 x} dx = \int \tan^2 x dx \\
 &= \int (\sec^2 x - 1) dx = \int \sec^2 x dx - \int dx \\
 &= \tan x - x + C.
 \end{aligned}$$

Example 17: Evaluate $\int \sin^{-1}(\cos x) dx$.

Solution: $\int \sin^{-1}(\cos x) dx = \int \sin^{-1} \left\{ \sin \left(\frac{\pi}{2} - x \right) \right\} dx$

$$= \int \left(\frac{\pi}{2} - x \right) dx = \frac{\pi}{2} \cdot \int dx - \int x dx = \frac{\pi x}{2} - \frac{x^2}{2} + C.$$

13.4 SUMMARY

In this unit, we focused on the concept and techniques of indefinite integration, which involves finding the antiderivative of a function. Indefinite integrals represent a family of functions and include an arbitrary constant of integration.

Key points covered include:

- **Definition of Indefinite Integrals** – Understanding integration as the reverse process of differentiation.
- **Basic Formulae** – Applying standard rules for integrating power functions, exponential functions, and trigonometric functions.

- **Constant of Integration (C)** – Recognizing the role of the integration constant in representing an entire family of solutions.
- **Properties and Linearity** – Exploring the linearity of integration and how to handle the sum, difference, and constant multiples of functions.

Mastering indefinite integrals is essential for understanding more advanced topics in calculus and applying integration techniques to solve diverse mathematical and real-world problems.

13.5 GLOSSARY

Integration: The process of finding an antiderivative or accumulated quantity.

Antiderivative: A function whose derivative is the given function.

Indefinite Integral: A family of antiderivatives containing an arbitrary constant.

Definite Integral: An integral evaluated between specified limits.

Constant of Integration: An arbitrary constant added to an indefinite integral.

Integrand: The function being integrated.

Integral Sign: The symbol $\int$ used to denote integration.

Fundamental Theorem of Calculus: A theorem connecting differentiation and definite integration.

13.6 EXERCISES

5. Integrate $3x + 4x^2$.
6. Integrate $\sec x \tan x - 5 \operatorname{cosec}^2 x$.
7. Integrate $\frac{2 \cos x}{3 \sin^2 x} + 1$.
8. Integrate $\frac{a}{x^2} + \frac{b}{x} + c$.

9. Evaluate $\int \frac{4x+6}{x} dx$.
10. Evaluate $\int (e^x + 2 \sin x - 3 \cos x) dx$.
11. Integrate $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
12. Integrate $10^x + 3e^x + x^3$.

13.7 SUGGESTED READINGS

1. T. M. Apostol, Calculus Vol I, Wiley & Sons (Asia) Pvt. Ltd, 1991.
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UNIT 14 METHOD OF INTEGRATION BY SUBSTITUTION

STRUCTURE

- 14.1. Introduction
- 14.2. Objectives
- 14.3. Method of integration by substitution
- 14.4. Summary
- 14.5. Glossary
- 14.6. Exercises
- 14.7. Suggested Readings

14.1 INTRODUCTION

In the realm of calculus, integration serves as a powerful tool for finding areas under curves, determining volumes of irregular shapes, and solving various real-world problems. Among the arsenal of integration techniques, the method of integration by substitution stands out as a versatile and fundamental approach. This unit serves as an entry point into understanding the principles and applications of integration by substitution.

14.2 OBJECTIVES

After completing this unit, students will be able to

- Understand the concept of substitution and its role in integration.
- Learn how to identify appropriate substitutions for different types of integrals.
- Master the technique of integration by substitution through step-by-step examples.
- Apply integration by substitution to solve a variety of integrals involving trigonometric, exponential, and other functions.

- Explore advanced applications of integration by substitution in calculus and beyond.

14.3 METHOD OF INTEGRATION BY SUBSTITUTION

If we have to evaluate an integral of the type $\int f\{\phi(x)\} \cdot \phi'(x)dx$ then we put $\phi(x) = t$ and $\phi'(x)dx = dt$. With this substitution, the integrand becomes easily integrable.

Substitution Method I: When the integrand is of the form $f(ax + b)$, we put $(ax + b) = t$ and $dx = \frac{1}{a} dt$.

Substitution Method II: When the integrand is of the form $x^{n-1} \cdot f(x^n)$, we put $x^n = t$ and $nx^{n-1}dx = dt$.

Substitution Method III: When the integrand is of the form $\{f(x)\}^n \cdot f'(x)$, we put $f(x) = t$ and $f'(x)dx = dt$.

Substitution Method IV: When the integrand is of the form $\frac{f'(x)}{f(x)}$, we put $f(x) = t$ and $f'(x)dx = dt$.

Example 1: $\int (ax + b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$, where $n \neq -1$.

Solution: Putting $ax + b = t$, we get $adx = dt$ or $dx = \frac{1}{a} dt$.

$$\therefore \int (ax + b)^n dx = \frac{1}{a} \int t^n dt = \frac{1}{a} \cdot \frac{t^{n+1}}{(n+1)} + C = \frac{(ax + b)^{n+1}}{a(n+1)} + C.$$

Example 2: $\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + C$

Solution: Put $(ax + b) = t$ so that $dx = \frac{1}{a} dt$.

$$\begin{aligned}
 \therefore \int \cos(ax+b)dx &= \frac{1}{a} \int \cos t dt \\
 &= \frac{1}{a} \sin t + C \\
 &= \frac{1}{a} \sin(ax+b) + C.
 \end{aligned}$$

Example 3: $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a} \cot(ax+b) + C$

Solution: Put $(ax+b) = t$ so that $dx = \frac{1}{a} dt$.

$$\begin{aligned}
 \therefore \int \operatorname{cosec}^2(ax+b)dx &= \frac{1}{a} \int \operatorname{cosec}^2 t dt \\
 &= -\frac{1}{a} \cot t + C = -\frac{1}{a} \cot(ax+b) + C.
 \end{aligned}$$

Example 4: Evaluate: $\int (3x+5)^7 dx$

Solution: Put $(3x+5) = t$ so that $3dx = dt$ or $dx = \frac{1}{3} dt$.

$$\therefore \int (3x+5)^7 dx = \frac{1}{3} \int t^7 dt = \frac{1}{3} \cdot \frac{t^8}{8} + C = \frac{(3x+5)^8}{24} + C.$$

Example 5: Evaluate $\int (4-9x)^5 dx$.

Solution: Put $(4-9x) = t$ so that $-9dx = dt$ or $dx = -\frac{1}{9} dt$.

$$\therefore \int (4-9x)^5 dx = -\frac{1}{9} \int t^5 dt = -\frac{1}{9} \cdot \frac{t^6}{6} + C = \frac{-(4-9x)^6}{54} + C.$$

Example 6: Evaluate $\int \frac{1}{(2-3x)^4} dx$

Solution: Put $(2-3x) = t$ so that $-3dx = dt$ or $dx = -\frac{1}{3} dt$.

$$\begin{aligned}\therefore \int \frac{1}{(2-3x)^4} dx &= -\frac{1}{3} \int \frac{1}{t^4} dt = -\frac{1}{3} \cdot \frac{1}{(-3t^3)} + C \\ &= \frac{1}{9(2-3x)^3} + C.\end{aligned}$$

Example 7: Evaluate $\int \sqrt{ax+b} dx$

Solution: Put $(ax+b) = t$ so that $adx = dt$.

$$\therefore \int \sqrt{ax+b} dt = \frac{1}{a} \int \sqrt{t} dt = \frac{2}{3a} t^{3/2} + C = \frac{2(ax+b)^{3/2}}{3a} + C.$$

Example 8: Evaluate $\int \cos 2x dx$.

Solution: Put $2x = t$ so that $2dx = dt$ or $dx = \frac{1}{2} dt$.

$$\therefore \int \cos 2x dx = \frac{1}{2} \int \cos t dt = \frac{1}{2} \sin t + C = \frac{1}{2} \sin 2x + C.$$

Example 9: Evaluate $\int e^{(5x+3)} dx$

Solution: Put $(5x+3) = t$ so that $5dx = dt$ or $dx = \frac{1}{5} dt$.

$$\therefore \int e^{(5x+3)} dx = \frac{1}{5} \int e^t dt = \frac{1}{5} \cdot e^t + C = \frac{1}{5} e^{(5x+3)} + C.$$

Example 10: Evaluate $\int \sec^2 (3x+5) dx$

Solution: Put $(3x+5) = t$ so that $3dx = dt$ or $dx = \frac{1}{3} dt$.

$$\begin{aligned}\therefore \int \sec^2 (3x+5) dx &= \frac{1}{3} \int \sec^2 t dt = \frac{1}{3} \tan t + C \\ &= \frac{1}{3} \tan (3x+5) + C.\end{aligned}$$

Example 11: Evaluate $\int \sin^3 x dx$

Solution: We know that $\sin 3x = 3\sin x - 4\sin^3 x$.

$$\therefore \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x).$$

$$\begin{aligned}\text{So, } \int \sin^3 x dx &= \int \left(\frac{3}{4} \sin x - \frac{1}{4} \sin 3x \right) dx \\ &= \frac{3}{4} \int \sin x dx - \frac{1}{4} \int \sin 3x dx \\ &= \frac{3}{4} (-\cos x) - \frac{1}{4} \cdot \frac{(-\cos 3x)}{3} + C \\ &= -\frac{3}{4} \cos x + \frac{\cos 3x}{12} + C.\end{aligned}$$

Example 12: Evaluate $\int \frac{\log x}{x} dx$

Solution: Put $\log x = t$ so that $\frac{1}{x} dx = dt$.

$$\therefore \int \frac{\log x}{x} dx = \int t dt = \frac{1}{2} t^2 + C = \frac{1}{2} (\log x)^2 + C.$$

Example 13: Evaluate $\int \frac{\sec^2 (\log x)}{x} dx$

Solution: Put $\log x = t$ so that $\frac{1}{x} dx = dt$.

$$\therefore \int \frac{\sec^2 (\log x)}{x} dx = \int \sec^2 t dt = \tan t + C = \tan (\log x) + C.$$

Example 14: Evaluate $\int \frac{e^{\tan^{-1} x}}{(1+x^2)} dx$

Solution: Put $\tan^{-1} x = t$ so that $\frac{1}{(1+x^2)} dx = dt$.

$$\therefore \int \frac{e^{\tan^{-1} x}}{(1+x^2)} dx = \int e^t dt = e^t + C = e^{\tan^{-1} x} + C.$$

Example 15: Evaluate $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

Solution: Put $\sqrt{x} = t$ so that $\frac{1}{2} x^{-1/2} dx = dt$ or $\frac{1}{\sqrt{x}} dx = 2dt$.

$$\begin{aligned} \therefore \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx &= 2 \int \sin t dt \\ &= 2(-\cos t) + C = -2\cos t + C = -2\cos \sqrt{x} + C. \end{aligned}$$

Example 16: Evaluate $\int \cos^3 x \sin x dx$

Solution: Put $\cos x = t$ so that $\sin x dx = -dt$.

$$\therefore \int \cos^3 x \sin x dx = - \int t^3 dt = -\frac{t^4}{4} + C = -\frac{1}{4} \cos^4 x + C.$$

Example 17: Evaluate $\int (\sqrt{\sin x}) \cos x dx$

(ii) Put $\sin x = t$ so that $\cos x dx = dt$.

$$\therefore \int (\sqrt{\sin x}) \cos x dx = \int \sqrt{t} dt = \frac{2}{3} t^{3/2} + C = \frac{2}{3} (\sin x)^{3/2} + C.$$

Example 18: Evaluate $\int \frac{\operatorname{cosec}^2 x}{(1+\cot x)} dx$

Solution: Put $(1 + \cot x) = t$ so that $-\operatorname{cosec}^2 x dx = dt$.

$$\begin{aligned} \therefore \int \frac{\operatorname{cosec}^2 x}{(1+\cot x)} dx &= - \int \frac{1}{t} dt \\ &= -\log t + C = -\log |(1+\cot x)| + C. \end{aligned}$$

Example 19: Evaluate $\int \frac{\sin x}{(3+4\cos x)^2} dx$

Solution: Put $(3 + 4\cos x) = t$ so that $-4\sin x dx = dt$.

$$\therefore \int \frac{\sin x}{(3 + 4\cos x)^2} dx = -\frac{1}{4} \int \frac{1}{t^2} dt = \frac{1}{4t} + C = \frac{1}{4(3 + 4\cos x)} + C.$$

Example 20: Evaluate $\int \frac{2x}{(2x+1)^2} dx$

Solution: Put $(2x + 1) = t$ so that $2x = (t - 1)$ and $dx = \frac{1}{2} dt$.

$$\begin{aligned} \therefore \int \frac{2x}{(2x+1)^2} dx &= \frac{1}{2} \int \frac{(t-1)}{t^2} dt \\ &= \frac{1}{2} \int \frac{1}{t} dt - \frac{1}{2} \int \frac{1}{t^2} dt = \frac{1}{2} \log |t| + \frac{1}{2t} + C \\ &= \frac{1}{2} \log |(2x+1)| + \frac{1}{2(2x+1)} + C. \end{aligned}$$

Example 21: Evaluate $\int \frac{(2+3x)}{(3-2x)} dx$

Solution: Put $(3 - 2x) = t$ so that $x = \left(\frac{3-t}{2}\right)$ and $dx = -\frac{1}{2} dt$.

$$\begin{aligned} \therefore \int \frac{(2+3x)}{(3-2x)} dx &= -\frac{1}{2} \int \frac{\left[2 + \left(\frac{9-3t}{2}\right)\right]}{t} dt = -\frac{1}{4} \int \frac{(13-3t)}{t} dt \\ &= -\frac{13}{4} \int \frac{1}{t} dt + \frac{3}{4} \int dt = -\frac{13}{4} \log |t| + \frac{3}{4} t + C \\ &= -\frac{13}{4} \log |(3-2x)| + \frac{3}{4} (3-2x) + C. \end{aligned}$$

Example 22: Evaluate $\int \frac{3x^2}{(1+x^6)} dx$

Solution: Put $x^3 = t$ so that $3x^2 dx = dt$.

$$\therefore \int \frac{3x^2}{(1+x^6)} dx = \int \frac{dt}{(1+t^2)} = \tan^{-1} t + C = \tan^{-1} x^3 + C.$$

Example 23: Evaluate $\int \frac{x^3}{(x^2+1)^3} dx$

Solution: Put $(x^2 + 1) = t$ so that $x^2 = (t - 1)$ and $x dx = \frac{1}{2} dt$.

$$\begin{aligned}\therefore \int \frac{x^3}{(x^2 + 1)^3} dx &= \int \frac{x^2 \cdot x}{(x^2 + 1)^3} dx \\&= \frac{1}{2} \int \frac{(t - 1)}{t^3} dt = \frac{1}{2} \int \frac{1}{t^2} dt - \frac{1}{2} \int \frac{1}{t^3} dt \\&= \frac{-1}{2t} + \frac{1}{4t^2} + C = \frac{-1}{2(x^2 + 1)} + \frac{1}{4(x^2 + 1)^2} + C \\&= \frac{-(1 + 2x^2)}{4(x^2 + 1)^2} + C.\end{aligned}$$

Example 24: $\frac{x^8}{(1-x^3)^{1/3}} dx$

Solution: Put $(1 - x^3) = t$ so that $x^3 = (1 - t)$ and $x^2 dx = -\frac{1}{3} dt$.

$$\begin{aligned}\therefore \int \frac{x^8}{(1 - x^3)^{1/3}} dx &= \int \frac{x^6 \cdot x^2}{(1 - x^3)^{1/3}} dx \\&= -\frac{1}{3} \int \frac{(1 - t)^2}{t^{1/3}} dt = -\frac{1}{3} \int \frac{(1 + t^2 - 2t)}{t^{1/3}} dt \\&= -\frac{1}{3} \int t^{-1/3} dt - \frac{1}{3} \int t^{5/3} dt + \frac{2}{3} \int t^{2/3} dt \\&= -\frac{1}{2} t^{2/3} - \frac{1}{8} t^{8/3} + \frac{2}{5} t^{5/3} + C \\&= -\frac{1}{2} (1 - x^3)^{2/3} - \frac{1}{8} (1 - x^3)^{8/3} \\&\quad + \frac{2}{5} (1 - x^3)^{5/3} + C.\end{aligned}$$

Example 25: Evaluate $\int \frac{dx}{x \cdot \sqrt{x^6 - 1}}$.

Solution: Put $x^3 = t$ so that $3x^2 dx = dt$ or $x^2 dx = \frac{1}{3} dt$.

$$\therefore \int \frac{dx}{x \cdot \sqrt{x^6 - 1}} = \int \frac{x^2}{x^3 \cdot \sqrt{x^6 - 1}} dx$$

[multiplying numerator and denominator by x^2]

$$= \frac{1}{3} \int \frac{1}{t\sqrt{t^2-1}} dt = \frac{1}{3} \sec^{-1} t + C = \frac{1}{3} \sec^{-1} x^3 + C.$$

Example 26: Evaluate $\int \frac{1}{(\sqrt{x}+x)} dx$

Solution: $\int \frac{1}{(\sqrt{x}+x)} dx = \int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$

Now, put $(1 + \sqrt{x}) = t$ so that $\frac{1}{\sqrt{x}} dx = 2dt$.

$$\begin{aligned} \therefore \int \frac{1}{(\sqrt{x} + x)} dx &= \int \frac{1}{\sqrt{x}(1 + \sqrt{x})} dx \\ &= 2 \int \frac{1}{t} dt = 2 \log |t| + C = 2 \log |(1 + \sqrt{x})| + C. \end{aligned}$$

Example 27: Evaluate $\int \frac{(x-1)}{\sqrt{x+4}} dx$

Solution: Put $(x + 4) = t^2$ so that $x = (t^2 - 4)$ and $dx = 2t dt$.

$$\begin{aligned} \therefore \int \frac{(x-1)}{\sqrt{x+4}} dx &= 2 \int \frac{(t^2-5)t}{t} dt \\ &= 2 \int t^2 dt - 10 \int dt = \frac{2t^3}{3} - 10t + C \\ &= \frac{2}{3}(x+4)^{3/2} - 10(x+4)^{1/2} + C. \end{aligned}$$

Example 28: Evaluate $\int x\sqrt{x+2} dx$

Solution: Put $(x + 2) = t^2$ so that $x = (t^2 - 2)$ and $dx = 2t dt$.

$$\begin{aligned} \therefore \int x\sqrt{x+2} dx &= \int (t^2 - 2)2t dt = 2 \int t^4 dt - 4 \int t^2 dt \\ &= \frac{2t^5}{5} - \frac{4t^3}{3} + C = \frac{2(x+2)^{5/2}}{5} - \frac{4(x+2)^{3/2}}{3} + C. \end{aligned}$$

Example 29: Evaluate $\int (4x + 2)\sqrt{x^2 + x + 1}dx$

Solution: Put $(x^2 + x + 1) = t$ so that $(2x + 1)dx = dt$.

$$\begin{aligned}\therefore \int (4x + 2)(\sqrt{x^2 + x + 1})dx &= 2 \int \sqrt{t}dt \\ &= \frac{4}{3}t^{3/2} + C = \frac{4}{3}(x^2 + x + 1)^{3/2} + C.\end{aligned}$$

Example 30: Evaluate $\int \frac{(4x+3)}{\sqrt{2x^2+3x+1}}dx$

Solution: Put $(2x^2 + 3x + 1) = t$ so that $(4x + 3)dx = dt$.

$$\therefore \int \frac{(4x + 3)}{\sqrt{2x^2 + 3x + 1}}dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C = 2\sqrt{2x^2 + 3x + 1} + C.$$

Example 31: Evaluate $\int \frac{(2x+5)}{(x^2+5x+9)}dx$

Solution: Put $(x^2 + 5x + 9) = t$ so that $(2x + 5)dx = dt$.

$$\begin{aligned}\therefore \int \frac{(2x + 5)}{(x^2 + 5x + 9)}dx &= \int \frac{1}{t}dt = \log |t| + C \\ &= \log |(x^2 + 5x + 9)| + C.\end{aligned}$$

Example 32: Evaluate $\int \frac{(6x-7)}{(3x^2-7x+5)^2}dx$

Solution: Put $(3x^2 - 7x + 5) = t$ so that $(6x - 7)dx = dt$.

$$\therefore \int \frac{(6x - 7)}{(3x^2 - 7x + 5)^2}dx = \int \frac{1}{t^2}dt = -\frac{1}{t} + C = \frac{-1}{(3x^2 - 7x + 5)} + C.$$

Example 33: Evaluate $\int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}dx$

Solution: Put $(\cos x + \sin x) = t$ so that $(\cos x - \sin x)dx = dt$.

$$\begin{aligned}\therefore \int \frac{(\cos x - \sin x)}{(\cos x + \sin x)} dx &= \int \frac{1}{t} dt \\ &= \log |t| + C = \log |(\cos x + \sin x)| + C.\end{aligned}$$

Check Your Progress

1. Evaluate: $\int 2x(x^2 + 1)^5 dx$ using substitution.

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2. Evaluate: $\int e^{(3x)} dx$ using a suitable substitution.

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3. Evaluate: $\int x/(x^2 + 4) dx$ using substitution.

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Example 34: Evaluate $\int \frac{\sec x}{\log (\sec x + \tan x)} dx$

Solution: Put $\log (\sec x + \tan x) = t$.

Then, on differentiation, we get

$$\begin{aligned} & \frac{1}{(\sec x + \tan x)} \cdot (\sec x \tan x + \sec^2 x) dx = dt \\ & \text{or } \sec x dx = dt \\ & \therefore \int \frac{\sec x}{\log(\sec x + \tan x)} dx = \int \frac{1}{t} dt \\ & = \log |t| + C = \log |\log(\sec x + \tan x)| + C. \end{aligned}$$

Example 35: Evaluate $\int \frac{\sin 2x}{(a^2 \sin^2 x + b^2 \cos^2 x)} dx$.

Solution: Put $(a^2 \sin^2 x + b^2 \cos^2 x) = t$ so that

$$\begin{aligned} 2(a^2 - b^2) \sin x \cos x dx &= dt \Leftrightarrow \sin 2x dx = \frac{dt}{(a^2 - b^2)} \\ \therefore I &= \int \frac{\sin 2x}{(a^2 \sin^2 x + b^2 \cos^2 x)} dx = \int \frac{dt}{(a^2 - b^2)t} \\ &= \frac{1}{(a^2 - b^2)} \log |t| + C \\ &= \frac{1}{(a^2 - b^2)} \log (a^2 \sin^2 x + b^2 \cos^2 x) + C. \end{aligned}$$

14.4 SUMMARY

So, in this first unit we have learned about the

- the concept of substitution and its role in integration.
- how to identify appropriate substitutions for different types of integrals.
- the technique of integration by substitution through step-by-step examples.
- integration by substitution to solve a variety of integrals involving trigonometric, exponential, and other functions.

- advanced applications of integration by substitution in calculus and beyond.

14.5 GLOSSARY

Substitution: Replacing part of an integrand with a new variable.

Change of Variable: The transformation from the original variable to a new variable.

Integrand: The expression to be integrated.

Substitution Variable: The new variable introduced to simplify an integral.

Differential: An expression such as du representing an infinitesimal change.

Composite Function: A function formed by applying one function to another.

Reverse Chain Rule: A method of integration based on the chain rule of differentiation.

Simplification: The process of making an integral easier to evaluate.

14.6 EXERCISES

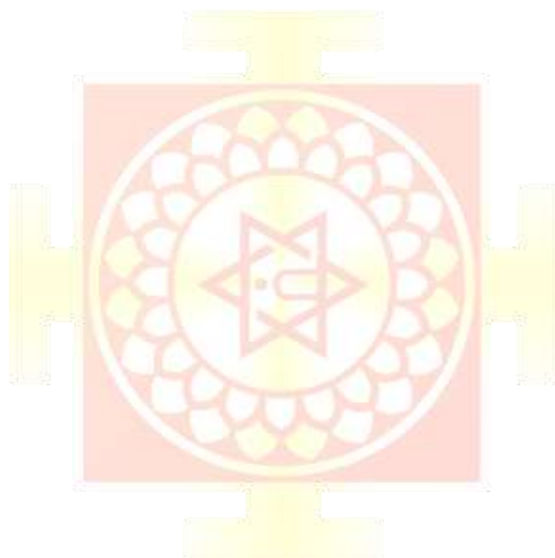
1. Evaluate $\int \frac{1}{c^2 + b^2 x^2} dx$.
2. Evaluate $\int e^x \cos e^x dx$.
3. Evaluate $\int \sin^2 x \cos x dx$.
4. Evaluate $\int \frac{4x^3}{1+x^8} dx$.
5. Evaluate $-\int \frac{\cos^{-1} x}{\sqrt{1-x^2}} dx$.
6. Evaluate $\int 20^{5x} dx$.
7. Evaluate $\int \frac{\operatorname{cosec} x^2}{1+\cot x} dx$.
8. Evaluate $\int \frac{1}{(1+x^2) \tan^{-1} x} dx$.

9. Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

10. Evaluate $\int \frac{\sin x \cos x}{a \cos^2 x + b \sin^2 x} dx$

14.7 SUGGESTED READINGS

1. T. M. Apostol, Calculus Vol I, Wiley & Sons (Asia) Pvt. Ltd, 1991.
2. Shanti Narayan, Integral Calculus, S. Chand, New Delhi, 2010.



UNIT 15 METHOD OF INTEGRATION BY PARTS

STRUCTURE

- | | |
|-------|--------------------------------|
| 15.1. | Introduction |
| 15.2. | Objectives |
| 15.3. | Method of integration by parts |
| 15.4. | Summary |
| 15.5. | Glossary |
| 15.6. | Exercises |
| 15.7. | Suggested Readings |
-

15.1 INTRODUCTION

In the broad field of calculus, integration by parts is an essential technique for tackling integrals that cannot be easily solved using other methods. This unit provides an introduction to the integration by parts method, offering learners a thorough understanding of its principles, uses, and limitations.

Integration by parts is a valuable tool for breaking down complex integrals, allowing mathematicians, scientists, and engineers to solve a wide range of problems in various fields. By the end of this unit, learners will have a strong command of integration by parts, empowering them to approach challenging integrals with confidence and expertise.

15.2 OBJECTIVES

After completing this unit, students will be able to

- Understand the concept of integration by parts and its significance in calculus.

- Learn the derivation of the integration by parts formula and its intuitive interpretation.
- Master the technique of integration by parts through guided examples and practice problems.
- Explore strategies for choosing appropriate functions to apply integration by parts effectively.
- Apply integration by parts to solve a variety of integrals, including those involving logarithmic, exponential, algebraic, and trigonometric functions.

15.3 METHOD OF INTEGRATION BY PARTS

Let $u = f(x)$ and $w = g(x)$ be two differentiable functions of x .

$$\begin{aligned}\therefore \frac{d}{dx}(uw) &= u \frac{dw}{dx} + w \frac{du}{dx} \text{ or } u \frac{dw}{dx} = \frac{d}{dx}(uw) - w \frac{du}{dx} \\ \therefore \int \left(u \frac{dw}{dx} \right) dx &= \int \left(\frac{d}{dx}(uw) \right) dx - \int \left(w \frac{du}{dx} \right) dx \quad (1)\end{aligned}$$

$$\frac{dw}{dx} = v \Rightarrow w = \int v dx$$

$$\therefore (1) \Rightarrow \int u v dx = \int \left(\frac{d}{dx}(u \int v dx) \right) dx - \int \left(\frac{du}{dx} \int v dx \right) dx$$

$$\therefore \int u v dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx.$$

In words: Integral of the product of two functions

= 1st function $\times$ Integral of 2nd - Integral of [Diff. coeff. of 1st $\times$ Integral of 2nd].

From the above formula, it is clear that the success of the method of integration by parts depends on the choice of first function. For the choice of first function, the following rules are used:

Rule (i) If the integrand contains both functions integrable, take that function as the first which can be finished by repeated differentiation and the other as the second.

For example, in $\int x^2 \sin x dx$ or in $\int x^2 e^x dx$, take x^2 as the first function and the other as the second.

Rule (ii) If the integrand contains one un integrable, eg., inverse circular function or a logarithmic function, take integrable as the first function and other as second. For example, in $\int x^2 \log x dx$, $\int x \sin^{-1} x dx$, take $\log x$, $\sin^{-1} x$ the first function and other as the second respectively.

Rule (iii) If the integrand contains only one integrable function, then take the given function as the first function and unity as the second. For example, in $\int \tan^{-1} x dx$
C. $\log(x^2 + a^2) dx$, take $\tan^{-1} x$, $\log(x^2 + a^2)$ as the first function and unity as the second function respectively.

Rule (iv) If the integrand contains both the functions integrable and none can be finished by repeated differentiation, then take any one as the first function and other as the second one. Repeat the rule of integration by parts.

Remark 1. If the integrand is the product of two functions of different type then their order is determined by the word ILATE where the letters **I**, **L**, **A**, **T** and **E** stands respectively for inverse circular function, logarithmic function, algebraic function, trigonometric function and exponential function. In the integrand, that function is taken as the first function which comes first in the word ILATE.

Remark 2. Sometimes, the formula of integration by parts is required to be applied more than once in evaluating the same integral. For example, in evaluating $\int x^2 e^x dx$, the formula for integration by parts would be used twice.

Integration by Parts

Theorem: If u and v are two functions of x then

$$\int (uv)dx = \left[u \cdot \int vdx \right] - \int \left\{ \frac{du}{dx} \cdot \int vdx \right\} dx.$$

Proof: For any two functions $f_1(x)$ and $f_2(x)$, we have

$$\begin{aligned} \frac{d}{dx} [f_1(x) \cdot f_2(x)] &= f_1(x) \cdot f_2'(x) + f_2(x) \cdot f_1'(x). \\ \therefore \int \{f_1(x) \cdot f_2'(x) + f_2(x) \cdot f_1'(x)\} dx &= f_1(x) \cdot f_2(x) \\ \text{or } \int f_1(x) \cdot f_2'(x) dx + \int f_2(x) \cdot f_1'(x) dx &= f_1(x) \cdot f_2(x) \\ \text{or } \int f_1(x) \cdot f_2'(x) dx &= f_1(x) \cdot f_2(x) - \int f_2(x) \cdot f_1'(x) dx. \end{aligned}$$

Let $f_1(x) = u$ and $f_2'(x) = v$ so that $f_2(x) = \int vdx$.

$$\therefore \int (uv)dx = u \cdot \int vdx - \int \left\{ \frac{du}{dx} \cdot \int vdx \right\} dx.$$

We can express this result as given below:

Integral of product of two functions

$$\begin{aligned} &= (1 \text{ st function}) \times (\text{integral of 2 nd}) \\ &\quad - \int \{(\text{derivative of 1 st}) \times (\text{integral of 2 nd})\} dx. \end{aligned}$$

REMARKS (i) If the integrand is of the form $f(x)x^n$, we consider x^n as the first function and $f(x)$ as the second function.

(ii) If the integrand contains a logarithmic or an inverse trigonometric function, we take it as the first function. In all such cases, if the second function is not given, we take it as 1 .

Example 1: Evaluate $\int x \sec^2 x \, dx$

Solution: Integrating by parts, taking x as the first function, we get

$$\begin{aligned} \int x \sec^2 x \, dx &= x \cdot \int \sec^2 x \, dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \sec^2 x \, dx \right\} dx \\ &= x \tan x - \int 1 \cdot \tan x \, dx = x \tan x + \log |\cos x| + C \end{aligned}$$

Example 2: Evaluate $\int x \sin 2x \, dx$

Solution: Integrating by parts, taking x as the first function, we get

$$\begin{aligned} \int x \sin 2x \, dx &= x \cdot \int \sin 2x \, dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \sin 2x \, dx \right\} dx \\ &= x \cdot \left(\frac{-\cos 2x}{2} \right) - \int 1 \cdot \left(\frac{-\cos 2x}{2} \right) dx \\ &= \frac{-x \cos 2x}{2} + \frac{1}{2} \int \cos 2x \, dx \\ &= \frac{-x \cos 2x}{2} + \frac{1}{2} \cdot \frac{\sin 2x}{2} + C \\ &= \frac{-x \cos 2x}{2} + \frac{1}{4} \sin 2x + C \end{aligned}$$

Example 3: Evaluate $\int x^n \log x \, dx$.

Solution: Integrating by parts, taking $\log x$ as the first function, we get

$$\begin{aligned}
 \int x^n \log x &= (\log x) \cdot \int x^n dx - \int \left\{ \frac{d}{dx} (\log x) \cdot \int x^n dx \right\} dx \\
 &= (\log x) \cdot \frac{x^{n+1}}{(n+1)} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{(n+1)} dx \\
 &= \frac{x^{n+1} \log x}{(n+1)} - \frac{1}{(n+1)} \int x^n dx \\
 &= \frac{x^{n+1} \log x}{(n+1)} - \frac{x^{n+1}}{(n+1)^2} + C.
 \end{aligned}$$

Example 4: Evaluate $\int x^2 \sin x dx$

Solution: Integrating by parts, taking x^2 as the first function, we get

$$\begin{aligned}
 \int x^2 \sin x dx &= x^2 \int \sin x dx - \int \left[\frac{d}{dx} (x^2) \cdot \int \sin x dx \right] dx \\
 &= x^2 (-\cos x) - \int 2x (-\cos x) dx \\
 &= -x^2 \cos x + 2 \int x \cos x dx \\
 &= -x^2 \cos x + 2 \left[x (\sin x) - \int \left\{ \frac{d}{dx} (x) \cdot \int \cos x dx \right\} dx \right] \\
 &= -x^2 \cos x + 2 \left[x \sin x - \int \sin x dx \right] \\
 &= -x^2 \cos x + 2 [x \sin x + \cos x] + C.
 \end{aligned}$$

Example 5: Evaluate $\int x \cos^2 x dx$

Solution: $\int x \cos^2 x dx = \int x \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{1}{2} \int x dx + \frac{1}{2} \int x \cos 2x dx$

$$= \frac{x^2}{4} + \frac{1}{2} \cdot \left[x \cdot \int \cos 2x dx - \int \left\{ \frac{d}{dx} (x) \cdot \int \cos 2x dx \right\} dx \right]$$

[integrating $x \cos 2x$ by parts]

$$\begin{aligned}
 &= \frac{x^2}{4} + \frac{1}{2} \left[\frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} dx \right] \\
 &= \frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{1}{4} \cdot \frac{(-\cos 2x)}{2} + C \\
 &= \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C.
 \end{aligned}$$

Check Your Progress

1. Evaluate: $\int x e^x dx$ using integration by parts.

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2. Evaluate: $\int x \ln x dx$ using integration by parts.

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3. Evaluate: $\int x^2 e^x dx$ using integration by parts, applying the method more than once if required.

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Example 6: Evaluate $\int \log x dx$

Solution: Integrating by parts, taking $\log x$ as the first function and 1 as the second function, we get

$$\begin{aligned}\int \log x dx &= \int (\log x \cdot 1) dx \\&= (\log x) \cdot \int 1 dx - \int \left\{ \frac{d}{dx} (\log x) \cdot \int 1 dx \right\} dx \\&= (\log x) \cdot x - \int \left(\frac{1}{x} \cdot x \right) dx = x \log x - \int dx \\&= x \log x - x + C = x(\log x - 1) + C.\end{aligned}$$

Example 7: Evaluate $\int \log (1 + x^2) dx$.

Solution: Integrating by parts, taking $\log (1 + x^2)$ as the first function and 1

as the second function, we get

$$\begin{aligned}\int \log (1 + x^2) dx &= \int \{ \log (1 + x^2) \cdot 1 \} dx \\&= \log (1 + x^2) \cdot \int 1 dx - \int \left[\frac{d}{dx} \{ \log (1 + x^2) \} \cdot \int 1 dx \right] dx \\&= \log (1 + x^2) \cdot x - \int \frac{2x}{(1 + x^2)} \cdot x dx \\&= x \log (1 + x^2) - 2 \int \frac{x^2}{(1 + x^2)} dx \\&= x \log (1 + x^2) - 2 \int \left(1 - \frac{1}{1 + x^2} \right) dx \\&= x \log (1 + x^2) - 2 \int dx + 2 \int \frac{dx}{(1 + x^2)} \\&= x \log (1 + x^2) - 2x + 2 \tan^{-1} x + C.\end{aligned}$$

Example 8: Evaluate $\int (\log x)^2 dx$.

Solution: Integrating by parts, taking $(\log x)^2$ as the first function and 1 as the second function, we get

$$\begin{aligned}\int (\log x)^2 dx &= \int \{(\log x)^2 \cdot 1\} dx \\&= (\log x)^2 \cdot \int 1 dx - \int \left\{ \frac{d}{dx} (\log x)^2 \cdot \int 1 dx \right\} dx \\&= x(\log x)^2 - \int \left(\frac{2 \log x}{x} \cdot x \right) dx \\&= x(\log x)^2 - 2 \int (\log x \cdot 1) dx \\&= x(\log x)^2 - 2 \left[(\log x) \int dx - \int \left\{ \frac{d}{dx} (\log x) \cdot \int dx \right\} dx \right] \\&= x(\log x)^2 - 2 \left[x \log x - \int \frac{1}{x} \cdot x dx \right] \\&= x(\log x)^2 - 2x \log x + 2x + C.\end{aligned}$$

Example 9: Evaluate $\int \frac{\log x}{x^2} dx$.

Solution: Integrating by parts, taking $\log x$ as the first function and $\frac{1}{x^2}$ as the second function, we get

$$\begin{aligned}\int \frac{\log x}{x^2} dx &= \int (\log x) \cdot \frac{1}{x^2} dx \\&= (\log x) \cdot \int \frac{1}{x^2} dx - \int \left\{ \frac{d}{dx} (\log x) \cdot \int \frac{1}{x^2} dx \right\} dx \\&= (\log x) \left(-\frac{1}{x} \right) - \int \frac{1}{x} \cdot \left(-\frac{1}{x} \right) dx = -\frac{\log x}{x} + \int \frac{1}{x^2} dx \\&= -\frac{\log x}{x} - \frac{1}{x} + C = \frac{-(\log x + 1)}{x} + C.\end{aligned}$$

Example 10: Evaluate $\int e^{2x} \sin x dx$.

Solution: Integrating by parts, we get

$$\begin{aligned}\int e^{2x} \sin x dx &= \left(e^{2x} \cdot \int \sin x dx \right) - \int \left\{ \frac{d}{dx} (e^{2x}) \cdot \int \sin x dx \right\} dx \\&= e^{2x} \cdot (-\cos x) - 2 \int e^{2x} (-\cos x) dx \\&= -e^{2x} \cos x + 2 \int e^{2x} \cos x dx \\&= -e^{2x} \cos x + 2 \left[\left(e^{2x} \cdot \int \cos x dx \right) - \int \left\{ \frac{d}{dx} (e^{2x}) \cdot \int \cos x dx \right\} dx \right] \\&\quad \text{[integrating } e^{2x} \cos x \text{ by parts]} \\&\therefore 5 \int e^{2x} \sin x dx = e^{2x} (2 \sin x - \cos x) + C \\&\text{or } \int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C.\end{aligned}$$

Example 11: Evaluate $\int \left(\frac{x - \sin x}{1 - \cos x} \right) dx$.

$$\begin{aligned}\text{Solution: } \int \left(\frac{x - \sin x}{1 - \cos x} \right) dx &= \int \frac{x}{(1 - \cos x)} dx - \int \frac{\sin x}{(1 - \cos x)} dx \\&= \int \frac{x}{2 \sin^2 (x/2)} dx - \int \frac{2 \sin (x/2) \cos (x/2)}{2 \sin^2 (x/2)} dx \\&= \frac{1}{2} \int x \operatorname{cosec}^2 (x/2) dx - \int \cot (x/2) dx \\&= \frac{1}{2} \left[x \cdot \int \operatorname{cosec}^2 (x/2) dx - \int \left\{ \frac{d}{dx} (x) \cdot \int \operatorname{cosec}^2 (x/2) dx \right\} dx \right] - \int \cot \frac{x}{2} dx \\&= \left[x \cdot \left(-2 \cot \frac{x}{2} \right) - \int \left[1 \cdot \left(-2 \cot \frac{x}{2} \right) \right] dx \right] - \int \cot (x/2) dx + C \\&= -x \cot \frac{x}{2} + \int \cot \frac{x}{2} dx - \int \cot \frac{x}{2} dx + C = -x \cot \frac{x}{2} + C.\end{aligned}$$

Example 12: Evaluate $\int x \tan^{-1} x dx$.

Solution: Integrating by parts, taking $\tan^{-1} x$ as the first function, we get

$$\begin{aligned}
 \int x \tan^{-1} x \, dx &= (\tan^{-1} x) \cdot \int x \, dx - \int \left\{ \frac{d}{dx} (\tan^{-1} x) \cdot \int x \, dx \right\} dx \\
 &= (\tan^{-1} x) \cdot \frac{x^2}{2} - \int \frac{1}{(1+x^2)} \cdot \frac{x^2}{2} dx \\
 &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \text{ [on dividing } x^2 \text{ by } 1+x^2 \text{]} \\
 &= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{(1+x^2)} dx \\
 &= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C \\
 &= \frac{1}{2} (1+x^2) \tan^{-1} x - \frac{1}{2} x + C.
 \end{aligned}$$

Example 13: Evaluate $\int x^2 \sin^{-1} x \, dx$.

Solution: Integrating by parts, taking $\sin^{-1} x$ as the first function we get

$$\begin{aligned}
 \int x^2 \sin^{-1} x \, dx &= (\sin^{-1} x) \cdot \frac{x^3}{3} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^3}{3} dx \\
 &= \frac{x^3 \sin^{-1} x}{3} - \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx \\
 &= \frac{x^3 \sin^{-1} x}{3} - \frac{1}{3} \int \frac{x \cdot x^2}{\sqrt{1-x^2}} dx \\
 &= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3} \int \frac{t(1-t^2)}{t} dt, \text{ where } (1-x^2) = t^2 \\
 &= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3} \int dt - \frac{1}{3} \int t^2 dt \\
 &= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3} t - \frac{1}{9} t^3 + C \\
 &= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3} \sqrt{1-x^2} - \frac{1}{9} (1-x^2)^{3/2} + C.
 \end{aligned}$$

Example 14: Evaluate $\int \cos^{-1} x \, dx$

(ii) $\int \tan^{-1} x \, dx$

(iii) $\int \sec^{-1} x \, dx$

Solution: Put $\cos^{-1} x = t$ so that $x = \cos t$ and $dx = -\sin t dt$.

$$\begin{aligned}\therefore \int \cos^{-1} x dx &= -\int t \sin t dt \\ &= -[t \cdot (-\cos t) - \int 1 \cdot (-\cos t) dt] \\ &= t \cos t - \int \cos t dt = t \cos t - \sin t + C \\ &= x \cos^{-1} x - \sqrt{1-x^2} + C \\ &\quad \left[\because \cos t = x \Rightarrow \sin t = \sqrt{1-x^2} \right].\end{aligned}$$

[integrating by parts]

Example 15: Evaluate $\int \tan^{-1} x dx$

Solution: Put $\tan^{-1} x = t$ so that $x = \tan t$ and $dx = \sec^2 t dt$.

$$\begin{aligned}\therefore \int \tan^{-1} x dx &= \int t \sec^2 t dt \\ &= t \cdot \tan t - \int 1 \cdot \tan t dt \quad [\text{integrating by parts}] \\ &= t \cdot \tan t + \log |\cos t| + C \\ &= (\tan^{-1} x) \cdot x + \log \left| \frac{1}{\sqrt{1+x^2}} \right| + C \\ &\quad \left[\because \tan t = x \Rightarrow \cos t = \frac{1}{\sqrt{1+x^2}} \right] \\ &= x(\tan^{-1} x) - \frac{1}{2} \log |1+x^2| + C.\end{aligned}$$

Example 16: Evaluate $\int \sec^{-1} x dx$

Solution: Put $\sec^{-1} x = t$ so that $x = \sec t$ and $dx = \sec t \tan t dt$.

$$\begin{aligned}\therefore \int \sec^{-1} x dx &= \int t(\sec t \tan t) dt \\ &= t(\sec t) - \int 1 \cdot \sec t dt \quad [\text{integrating by parts}] \\ &= t(\sec t) - \log |\sec t + \tan t| + C \\ &= t(\sec t) - \log \left| \sec t + \sqrt{\sec^2 t - 1} \right| + C \\ &= x(\sec^{-1} x) - \log \left| x + \sqrt{x^2 - 1} \right| + C.\end{aligned}$$

Example 17: Evaluate $\int (\sin^{-1} x)^2 dx$.

Solution: Putting $x = \sin t$ and $dx = \cos t dt$, we get

$$\begin{aligned}\int (\sin^{-1} x)^2 dx &= \int t^2 \cos t dt \\&= t^2 \cdot (\sin t) - \int 2t(\sin t) dt \quad [\text{integrating by parts}] \\&= t^2 \sin t - 2 \left[t(-\cos t) - \int 1 \cdot (-\cos t) dt \right] \\&= t^2 \sin t + 2t \cos t - 2 \sin t + C \\&= x(\sin^{-1} x)^2 + 2(\sin^{-1} x)\sqrt{1-x^2} - 2x + C.\end{aligned}$$

Example 18: Evaluate $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$.

Solution: Put $x = \sin t$ so that $dx = \cos t dt$ and $t = \sin^{-1} x$.

$$\begin{aligned}\therefore \int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx &= \int \frac{t \cos t}{(1-\sin^2 t)^{3/2}} dt = \int \frac{t \cos t}{\cos^3 t} dt \\&= \int t \sec^2 t dt \\&= t \cdot (\tan t) - \int 1 \cdot \tan t dt \quad [\text{integrating by parts}] \\&= t \cdot (\tan t) + \log |\cos t| + C \\&= (\sin^{-1} x) \cdot \frac{x}{\sqrt{1-x^2}} + \log \left| \sqrt{1-x^2} \right| + C. \\&\left[\because \cos t = \sqrt{1-x^2} \text{ and } \tan t = \frac{x}{\sqrt{1-x^2}} \right] \\&= \frac{x(\sin^{-1} x)}{\sqrt{1-x^2}} + \frac{1}{2} \log |(1-x^2)| + C.\end{aligned}$$

Example 19: Evaluate $\int \sec^3 x dx$.

Solution: $\int \sec^3 x dx = \int \sec x \cdot \sec^2 x dx$

$$= \sec x \cdot (\tan x) - \int \sec x \tan x (\tan x) dx$$

[integrating by parts]

$$\begin{aligned} &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\ \therefore 2 \int \sec^3 x dx &= \sec x \tan x + \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + C \\ \text{or } \int \sec^3 x dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + C'. \end{aligned}$$

Example 20: Evaluate $\int \tan^{-1} \sqrt{x} dx$.

Solution: Put $\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}} dx = dt$ or $dx = 2t dt$.

$$\begin{aligned} \therefore \int \tan^{-1} \sqrt{x} dx &= 2 \int t (\tan^{-1} t) dt \\ &= 2 \left[(\tan^{-1} t) \cdot \frac{t^2}{2} - \int \left\{ \frac{1}{(1+t^2)} \cdot \frac{t^2}{2} \right\} dt \right] + C \\ &= t^2 (\tan^{-1} t) - \int \frac{t^2}{(1+t^2)} dt + C \\ &= t^2 (\tan^{-1} t) - \int \frac{[(1+t^2) - 1]}{(1+t^2)} dt + C \\ &= t^2 (\tan^{-1} t) - \int dt + \int \frac{1}{(1+t^2)} dt + C \\ &= t^2 (\tan^{-1} t) - t + \tan^{-1} t + C = (t^2 + 1) \tan^{-1} t - t + C \\ &= (x + 1) \tan^{-1} \sqrt{x} - \sqrt{x} + C. \end{aligned}$$

15.4 SUMMARY

So, in this first unit we have learned about the

- concept of integration by parts and its significance in calculus.
- derivation of the integration by parts formula and its intuitive interpretation.

- technique of integration by parts through guided examples and practice problems.
- choosing appropriate functions to apply integration by parts effectively.
- integration by parts to solve a variety of integrals, including those involving logarithmic, exponential, algebraic, and trigonometric functions.

15.5 GLOSSARY

Integration by Parts: A method of integration derived from the product rule.

Product Rule: A differentiation rule for the product of two functions.

First Function: The function selected as u in integration by parts.

Second Function: The differential selected as dv .

u : The chosen first function in the integration-by-parts formula.

dv : The chosen differential part to be integrated.

v : The antiderivative of dv .

LIATE: A common guideline for choosing u : Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential.

15.6 EXERCISES

1. Evaluate $\int x^5 e^x dx$.
2. Evaluate $\int x^4 \sin x \, dx$.
3. Evaluate $\int x^n \log x \, dx$.
4. Evaluate $\int (\log x)(\log x) \, dx$.
5. Evaluate $\int e^{3x} \cos 4x \, dx$.
6. Evaluate $\int \frac{x e^x}{(x+1)^2} dx$.

7. Evaluate $\int \frac{\log x}{(1+\log x)^2} dx$.
8. Evaluate $\int \frac{e^x(1+\sin x)}{1+\cos x} dx$.
9. Show that $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$.
10. Show that $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$.

15.7 SUGGESTED READINGS

1. T. M. Apostol, Calculus Vol I, Wiley & Sons (Asia) Pvt. Ltd, 1991.
2. Shanti Narayan, Integral Calculus, S. Chand, New Delhi, 2010.



UNIT 16 DEFINITE INTEGRATION AND ITS APPLICATION

STRUCTURE

- 16.1. Introduction
- 16.2. Objectives
- 16.3. Definite integration concept
- 16.4. Properties of definite integral
- 16.5. Solved example
- 16.6. Applications of Definite integration
- 16.7. Summary
- 16.8. Glossary
- 16.9. Exercises
- 16.10. Suggested Readings

16.1 INTRODUCTION

Definite integration is a fundamental concept in calculus that allows the calculation of the area under a curve between two specified points. This unit explores the concept, properties, and techniques of definite integration and how it can be applied in various fields such as physics, economics, and engineering. We will also cover practical examples and applications, enabling you to understand and use definite integrals to solve real-world problems. Mastering definite integration will provide essential tools for evaluating quantities such as total area, distance, and accumulated change.

16.2 OBJECTIVES

By the end of this unit, you will be able to:

- Understand the concept and definition of a definite integral.
- Apply the properties of definite integrals to simplify calculations.

- Solve various types of definite integrals using appropriate techniques.
- Use definite integration to compute areas under curves, total accumulated quantities, and solve real-world problems.
- Apply definite integration in different fields such as physics, economics, and engineering.

16.3 DEFINITE INTEGRATION CONCEPT

If $f(x)$ is a continuous and non-negative function over a closed interval $[a, b]$ then $\int_a^b f(x)dx$ is called the definite integral of $f(x)$ between the limits a and b ($b > a$).

If $\int f(x)dx = F(x) + c$, then $\int_a^b f(x)dx = [F(x) + c]_a^b = F(b) - F(a)$ is a definite value. Here, a is called the lower limit and b is called the upper limit and the interval $[a, b]$ is called the range of integration.

Remarks

- $\int_a^b f(x)dx$ represents the area bounded by the lines $x = a$ and $x = b$.
- If $F(b) - F(a)$ is not a definite value, then the integral $\int_a^b f(x)dx$ is indefinite.

16.4 PROPERTIES OF DEFINITE INTEGRALS

- 1 $\int_a^a f(x)dx = 0$.
- 2 The value of definite integral is independent of the variable of integration.

then

$$\int_a^b f(x)dx = \int_a^b f(u)du$$

i.e.,

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) = \int_a^b f(u)du$$

$$3 \quad \int_a^b f(x)dx = -\int_b^a f(x)dx.$$

$$4 \quad \int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx, \text{ where } a < c < b.$$

$$5 \quad \int_0^a f(a-x)dx = \int_0^a f(x)dx.$$

$$6 \quad \text{If } f(x) \text{ is an even function of } x, \text{ then } \int_{-a}^a f(x)dx = 2\int_0^a f(x)dx \\ \text{and if } f(x) \text{ is an odd function then } \int_{-a}^a f(x)dx = 0.$$

$$7 \quad \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x) \\ \text{and } \int_0^{2a} f(x)dx = 0 \text{ if } f(2a-x) = -f(x)$$

$$8 \quad \int_0^{na} f(x)dx = n\int_0^a f(x)dx \text{ if } f(x+ma) = f(x) \text{ for all integral} \\ \text{values of } m \text{ and } n \text{ is positive integer.}$$

Example 1. $\int_1^3 2x^3 dx = -\int_3^1 2x^3 dx$

$$\int_1^3 2x^3 dx = \left. \frac{1}{2} x^4 \right|_1^3 = \frac{1}{2} (3)^4 - \frac{1}{2} (1)^4 = 40$$

Checking this answer,

$$\int_3^1 2x^3 dx = \left. \frac{1}{2} x^4 \right|_3^1 = \frac{1}{2} (1)^4 - \frac{1}{2} (3)^4 = -40$$

Example 2. $\int_5^5 (2x+3)dx = 0$

Checking this answer,

$$\int_5^5 (2x+3)dx = [x^2 + 3x]_5^5 = [(5)^2 + 3(5)] - [(5)^2 + 3(5)] = 0$$

Example 3. $\int_0^4 6x dx = \int_0^3 6x dx + \int_3^4 6x dx$

$$\int_0^4 6x dx = 3x^2 \Big|_0^4 = 3(4)^2 - 3(0)^2 = 48$$

$$\int_0^3 6x dx = 3x^2 \Big|_0^3 = 3(3)^2 - 3(0)^2 = 27$$

$$\int_3^4 6x dx = 3x^2 \Big|_3^4 = 3(4)^2 - 3(3)^2 = 21$$

Checking this answer, $48 = 27 + 21$.

Check Your Progress

1. Evaluate: $\int_0^2 (3x^2 + 2x + 1) dx$.

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2. Find the area under the curve $y = 2x + 1$ between $x = 0$ and $x = 3$.

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.....

3. If marginal cost is $MC(q) = 4q + 10$, find the change in total cost when output increases from $q = 2$ to $q = 5$ using definite integration.

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16.5 SOLVED EXAMPLES

Example 1. Evaluate definite integrals $\int_1^4 10x dx$.

Solution. $\int_1^4 10x dx = 5x^2 \Big|_1^4 = 5(4)^2 - 5(1)^2 = 75$

Example 2. Evaluate definite integrals $\int_1^3 (4x^3 + 6x)dx$.

Solution. $\int_1^3 (4x^3 + 6x)dx = [x^4 + 3x^2]_1^3$
 $= [(3)^4 + 3(3)^2] - [(1)^4 + 3(1)^2] = 108 - 4$
 $= 104.$

Example 3. Evaluate definite integrals $\int_0^6 5x dx$.

Solution. $\int_0^6 5x dx = 2.5x^2|_0^6 = 2.5(6)^2 - 2.5(0)^2 = 90$

Example 4. Evaluate definite integrals $\int_1^{10} 3x^2 dx$.

Solution. $\int_1^{10} 3x^2 dx = x^3|_1^{10} = (10)^3 - (1)^3 = 999$

Example 5. Evaluate definite integrals $\int_1^{64} x^{-2/3} dx$

Solution. $\int_1^{64} x^{-2/3} dx = 3x^{1/3}|_1^{64} = 3\sqrt[3]{64} - 3\sqrt[3]{1} = 9$

Example 5. Evaluate definite integrals $\int_1^3 (x^3 + x + 6)dx$

Solution. $\int_1^3 (x^3 + x + 6)dx = \left(\frac{1}{4}x^4 + \frac{1}{2}x^2 + 6x\right)\Big|_1^3$
 $= \frac{1}{4}(3)^4 + \frac{1}{2}(3)^2 + 6(3) - \left[\frac{1}{4}(1)^4 + \frac{1}{2}(1)^2 + 6(1)\right] = 36$

Example 6. Evaluate definite integrals $\int_1^4 (x^{-1/2} + 3x^{1/2})dx$

Solution. $\int_1^4 \left(x^{-\frac{1}{2}} + 3x^{\frac{1}{2}}\right) dx = \left(2x^{\frac{1}{2}} + 2x^{\frac{3}{2}}\right)\Big|_1^4$
 $= 2\sqrt{4} + 2\sqrt{4^3} - \left(2\sqrt{1} + 2\sqrt{1^3}\right) = 16.$

Example 7. Evaluate definite integrals $\int_0^3 4e^{2x} dx$

Solution. $\int_0^3 4e^{2x} dx = 2e^{2x}|_0^3 = 2(e^{2(3)} - e^{2(0)}) = 2(403.4 - 1) =$
 $804.8.$

Example 8. Evaluate definite integrals $\int_0^{10} 2e^{-2x} dx$

Solution. $\int_0^{10} 2e^{-2x} dx = -e^{-2x} \Big|_0^{10} = -e^{-2(10)} - (-e^{-2(0)}) = -e^{-20} + e^0 = 1.$

Example 9. Evaluate $\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2} dx$

Solution. Put $\tan^{-1} x = t$, so that $\frac{1}{1+x^2} dx = dt$.

Also, when

$$x = 0, t = \tan^{-1} 0 = 0$$

$$x = 1, t = \tan^{-1} 1 = \frac{\pi}{4}$$

and when

$$\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2} dx = \int_0^{\pi/4} t^2 dt = \left[\frac{t^3}{3} \right]_0^{\pi/4} = \frac{1}{3} \times \frac{\pi^3}{64} = \frac{\pi^3}{192}$$

Example 10. Evaluate $\int_0^1 x^2 e^x dx$.

Solution. Using integrating by parts taking x^2 as Ist function we have

$$\begin{aligned} \int_0^1 x^2 e^x dx &= [x^2 e^x]_0^1 - \int_0^1 2x e^x dx \\ &= [x^2 e^x]_0^1 - 2 \left[x e^x - \int_0^1 e^x dx \right]_0^1 \\ &= [x^2 e^x]_0^1 - 2[x e^x - e^x]_0^1 \\ &= (1 \cdot e^1 - 0 \times e^0) - 2[(1 \cdot e^1 - e^1) - (0 - e^0)] \\ &= e - 2(0 + 1) = e - 2 \end{aligned}$$

Example 11. Evaluate $\int_0^{\pi/2} \cos x \cdot \sin^2 x dx$.

Solution. Putting $\sin x = t$ so that $\cos x dx = dt$.

also, when $x = 0, t = \sin 0 = 0$ and when $x = \frac{\pi}{2}, t = \sin \frac{\pi}{2} = 1.$

$$\int_0^{\pi/2} \cos x \cdot \sin^2 x dx = \int_0^1 t^2 dt = \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3} [t^3]_0^1 = \frac{1}{3} (1^3 - 0) = \frac{1}{3}$$

Example 12. Evaluate $\int_1^2 \frac{dx}{(x+3)(x+4)}$

Solution. Let $I = \int_1^2 \frac{dx}{(x+3)(x+4)}$ and $\frac{1}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

$$\Rightarrow \frac{1}{(x+3)(x+4)} = \frac{A(x+4) + B(x+3)}{(x+3)(x+4)}$$

$$\Rightarrow 1 = A(x+4) + B(x+3)$$

Putting $x = -4, -3$.

When

$$x = -4$$

$$\Rightarrow \frac{1}{B} = -B$$

$$B = -1$$

When

$$x = -3$$

$$\Rightarrow 1 = A$$

$$I = \int_1^2 \frac{1}{(x+3)(x+4)} dx = \int_1^2 \frac{dx}{x+3} - \int_1^2 \frac{dx}{x+4}$$

$$= [\log(x+3)]_1^2 - [\log(x+4)]_1^2$$

$$= [\log 5 - \log 4] - [\log 6 - \log 5]$$

$$= \log 5 - \log 4 - \log 6 + \log 5$$

$$= 2\log 5 - \log 4 - \log 6$$

$$= 2\log 5 - \log 24$$

$$= \log \frac{25}{24}$$

Example 13. Evaluate the following integrals

(i) $\int_0^{\pi/2} \log \tan x dx$

(ii) $\int_0^{\pi/2} \log (1 + \tan \theta) d\theta$

Solution. (i) Consider, $I = \int_0^{\pi/2} \log \tan x dx$.

Now,

$$I = \int_0^{\pi/2} \log \tan \left(\frac{\pi}{2} - x \right) dx \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$I = \int_0^{\pi/2} \log \cot x dx$$

On adding, we get $2I = \int_0^{\pi/2} \log \tan x dx + \int_0^{\pi/2} \log \cot x dx$

$$= \int_0^{\pi/2} \log (\tan x \cot x) dx = \int_0^{\pi/2} \log 1 = 0$$

Hence,

$$2I = 0 \Rightarrow I = 0$$

(ii)

Consider $I = \int_0^{\pi/4} \log (1 + \tan \theta) d\theta$

$$I = \int_0^{\pi/4} \log \left[1 + \tan \left(\frac{\pi}{4} - \theta \right) \right] d\theta \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$= \int_0^{\pi/4} \log \left[1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right] d\theta = \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan \theta} \right) d\theta$$

$$= \int_0^{\pi/4} [\log 2 - \log (1 + \tan \theta)] d\theta$$

$$= \int_0^{\pi/4} \log 2 d\theta - \int_0^{\pi/4} \log (1 + \tan \theta) d\theta = \frac{\pi}{4} \log 2 - I$$

So, $2I = \frac{\pi}{4} \log 2$

Hence $I = \frac{\pi}{8} \log 2$.

16.6 APPLICATIONS OF DEFINITE INTEGRATION

DETERMINATION OF LENGTHS (RECTIFICATION)

The process of finding the length of an arc of a curve between two given points is called rectification.

16.6.1. ARC FORMULAE FOR CARTESIAN EQUATIONS

To find the length of the arc of the curve $y = f(x)$ between the points, where $x = a, x = b$.

Or

To show that the length of the arc of the curve $y = f(x)$ between the points, where $x = a, x = b$ is given by

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

where y and $\frac{dy}{dx}$ are continuous and single-valued functions in the interval $a \leq x \leq b$ and the integrand does not change sign in this interval.

Example 1. Find the length of the arc of the curve $y = \log \frac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 2$.

Solution. The equation of the curve is $y = \log \frac{e^x - 1}{e^x + 1}$

$$y = \log(e^x - 1) - \log(e^x + 1)$$

$$\frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}$$

$$= e^x \left[\frac{1}{e^x - 1} - \frac{1}{e^x + 1} \right]$$

$$= e^x \left[\frac{e^x + 1 - e^x + 1}{e^{2x} - 1} \right] = \frac{2e^x}{e^{2x} - 1}$$

$\therefore$ Required length of arc from $x = 1$ to $x = 2$ is

$$\begin{aligned} &= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^2 \sqrt{1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}} dx = \\ &\int_1^2 \sqrt{\frac{(e^{2x} - 1)^2 + 4e^{2x}}{(e^{2x} - 1)^2}} dx = \int_1^2 \sqrt{\frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}} dx \end{aligned}$$

$$\begin{aligned}
&= \int_1^2 \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = |\log(e^x - e^{-x})| = \log(e^2 - e^{-2}) - \log(e - e^{-1}) \\
&= \log\left(e^2 - \frac{1}{e^2}\right) - \log\left(e - \frac{1}{e}\right) = \log\left(\frac{e^4 - 1}{e^2}\right) - \log\left(\frac{e^2 - 1}{e}\right) \\
&= \log\left(\frac{e^4 - 1}{e^2} \cdot \frac{e}{e^2 - 1}\right) = \log\left(\frac{e^2 + 1}{e}\right) = \log(e + e^{-1}).
\end{aligned}$$

16.6.2. ARC FORMULAE FOR PARAMETRIC EQUATIONS

To show that the length of the arc of the curve $x = f(t), y = \phi(t)$ between the points $t = a, t = b$ is given by

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Where x and y are continuous functions and single valued functions of t in the interval $a \leq t \leq b$.

Example 2. Find the length of the arc of the curve

$$x = e^\theta \sin \theta, y = e^\theta \cos \theta \text{ from } \theta = 0 \text{ to } \theta = \pi$$

Solution. The given curve is $x = e^\theta \sin \theta, y = e^\theta \cos \theta$

$$\frac{dx}{d\theta} = e^\theta \cdot \cos \theta + \sin \theta \cdot e^\theta = e^\theta (\cos \theta + \sin \theta)$$

$$\frac{dy}{d\theta} = e^\theta (-\sin \theta) + \cos \theta \cdot e^\theta = e^\theta (\cos \theta - \sin \theta)$$

$$\Rightarrow \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = e^{2\theta} [(\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2] = 2e^{2\theta}.$$

$$\begin{aligned}
\text{Required length of arc} &= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_0^{\frac{\pi}{2}} \sqrt{2e^{2\theta}} d\theta \\
&= \sqrt{2} \int_0^{\frac{\pi}{2}} e^\theta d\theta = \sqrt{2}(e^{\frac{\pi}{2}} - 1)
\end{aligned}$$

16.6.3. ARC FORMULAE FOR POLAR EQUATIONS

Show that the length of the arc of the curve $r = f(\theta)$ between the points $\theta = \alpha, \theta = \beta$ is

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Example 3. (i) Find the length of the arc of the equiangular spiral $r = a e^{\theta \cot \alpha}$ between the points for which the radii vectors are r_1 and r_2 .

(ii) Find the length of the arc of the spiral $r = a\theta$ the points whose radii vectors are r_1 and r_2 .

Solution. (i) The equation of the curve is $r = a e^{\theta \cot \alpha}$

$$\therefore \frac{dr}{d\theta} = a e^{\theta \cot \alpha} \cdot \cot \alpha = r \cot \alpha.$$

Required length of arc from $r = r_1$ to r_2 is

$$\begin{aligned} &= \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} dr. \\ &= \int_{r_1}^{r_2} \sqrt{1 + r^2 \cdot \frac{1}{r^2 \cot^2 \alpha}} dr. \\ &= \int_{r_1}^{r_2} \sqrt{1 + \tan^2 \alpha} dr = \int_{r_1}^{r_2} \sec \alpha dr = (r_2 - r_1) \sec \alpha \end{aligned}$$

DETERMINATION OF AREA (QUADRATURE)

16.6.4 AREA FORMULA FOR CARTESIAN EQUATIONS

1. The area bounded by the curve $y = f(x)$, the x-axis and ordinates $x = a, x = b$ is

$$\int_a^b y dx,$$

where $y = f(x)$ is a continuous single-valued function and y does not change sign in the interval $a \leq x \leq b$.

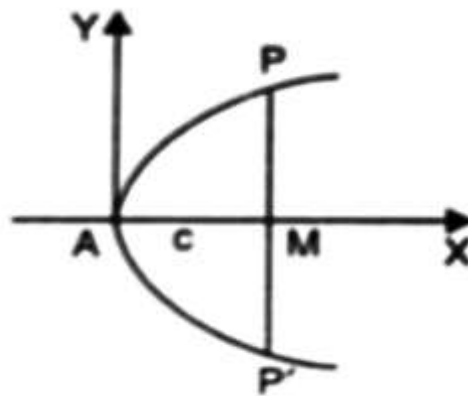
2. The area bounded by the curve $x = f(y)$, the x-axis and ordinates $y = c, y = d$ is

$$\int_c^d x \, dy,$$

where $x = f(y)$ is a continuous single-valued function and y does not change sign in the interval $c \leq y \leq d$.

Example 1. Show that the area cut off a parabola by any double ordinate is $2/3$ of the corresponding rectangle contained by the double ordinate and its distance from the vertex.

Solution. Let the equation of the parabola be $y^2 = 4ax$ and that of the double ordinate PP' be $x = c$.



The curve is symmetrical about the x-axis and for the part of the curve above the x-axis (taking + ve sign before the square root), we have $y = \sqrt{4ax}$

Area cut-off the parabola by the double ordinate $PMP = \text{area } PAP = 2 \text{ area } PAM$

$$= \int_0^c y \, dx = 2 \int_0^c \sqrt{4ax} \, dx = 4\sqrt{a} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right] = \frac{8}{3} \sqrt{a} c^{\frac{3}{2}}$$

Again $MP = \sqrt{4ax} = \sqrt{4ac} = 2\sqrt{a}\sqrt{c} \Rightarrow PP' = 2MP = 4\sqrt{a}\sqrt{c}.$

Area of the rectangle formed by the double ordinate PMP' and its distance from the vertex i.e., AM ,

$$= PP' \times AM = 4\sqrt{a}\sqrt{c} \times c = 4\sqrt{a}c^{\frac{3}{2}}$$

Area cut-off the parabola by the double ordinate $PP' = \frac{8}{3}\sqrt{a}c^{\frac{3}{2}} = \frac{2}{3}(4\sqrt{a}c^{\frac{3}{2}})$

$$= \frac{2}{3} \times \text{area of the rectangle formed by } PP' \text{ and } AM.$$

Hence proved.

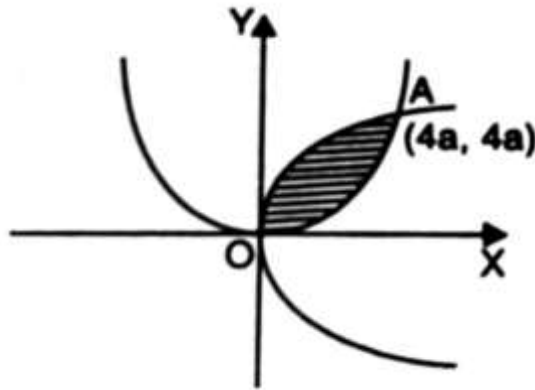
16.6.5 AREA BETWEEN TWO CURVES

The area bounded by the curves $y = f(x)$, $y = g(x)$ and the ordinates $x = a$, $x = b$ is

$$\int_a^b [y \text{ of upper curve} - y \text{ of lower curve}] dx$$

Example 2. Prove that the common area to the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $\frac{16}{3}a^2$.

Solution. To find the common area between the parabolas $x^2 = 4ay$ and $y^2 = 4ax$, we need to find the points of intersection of the two curves. Setting $x^2 = 4ay$ and $y^2 = 4ax$ and solving for x and y , we get $(0,0)$, $(4a, 4a)$.



Required common area to both the parabolas

$$\begin{aligned}
 &= \int_0^{4a} [y \text{ of upper curve} - y \text{ of lower curve}] dx \\
 &= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a} \right) dx \\
 &= \left[2\sqrt{a} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4a} \cdot \frac{x^3}{3} \right]_0^{4a} = \left[\frac{4\sqrt{a}}{3} x^{\frac{3}{2}} - \frac{1}{12a} x^3 \right]_0^{4a} \\
 &= \left[\frac{4\sqrt{a}}{3} (4a)^{\frac{3}{2}} - \frac{1}{12a} (4a)^3 \right] - [0 - 0] = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.
 \end{aligned}$$

16.6.6 AREA FORMULAE FOR PARAMETRIC EQUATIONS

(i) The area bounded by the curve $x = f(t)$ $y = \phi(t)$ the X-axis and the ordinates at the points where $t = a$ $t = b$ is $\int_a^b y \frac{dx}{dt} dt$.

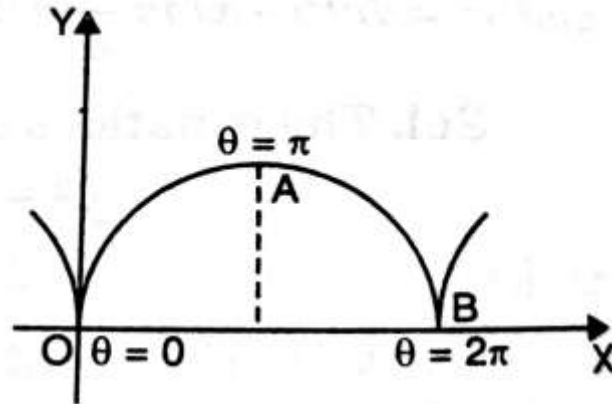
(ii) Similarly, the area bounded by the curve $x = f(t)$ $y = \phi(t)$, the y-axis and the abscissae at the points where $t = c, t = d$ is given by $\int_c^d x \frac{dy}{dt} dt$.

Example 3. Find the area enclosed between one arch of the cycloids

$x = a(\theta - \sin \theta)$ $y = a(1 - \cos \theta)$ and its base.

Solution. The parametric equations of the cycloid $x = a(\theta - \sin \theta)$ $y = a(1 - \cos \theta)$. The cycloid is symmetrical about the line $x = a\pi$ i.e., the line through the point where $\theta = \pi$ and perpendicular to x-axis, and the base is x-axis.

For the first half of the cycloid, θ varies from 0 to π .



Required area bounded by the cycloid and its base

= area under the cycloid

$$\begin{aligned}
 &= 2 \int_0^\pi y \frac{dx}{d\theta} d\theta = 2 \int_0^\pi a(1 - \cos \theta) a(1 - \cos \theta) d\theta \\
 &= 2a^2 \int_0^\pi (1 - \cos \theta)^2 d\theta = 2a^2 \int_0^\pi \left(2 \sin^2 \frac{\theta}{2}\right)^2 d\theta \\
 &= 8a^2 \int_0^\pi \sin^4 \frac{\theta}{2} \\
 &= 8a^2 \int_0^{\frac{\pi}{2}} \sin^4 t \cdot 2dt \\
 &= 16a^2 \int_0^{\frac{\pi}{2}} \sin^4 t dt = 16a^2 \cdot \frac{3.1}{4.2} \frac{\pi}{2} = 3\pi a^2.
 \end{aligned}$$

16.7 SUMMARY

So, in this first unit we have discussed the following concepts

- Definite integration concept
- Properties of definite integral
- Solved example
- Applications of Definite integration

16.8 GLOSSARY

Definite Integral: An integral with specified lower and upper limits.

Lower Limit: The starting limit of a definite integral.

Upper Limit: The ending limit of a definite integral.

Area Under Curve: The accumulated area represented by a definite integral.

Net Area: Area above a curve offset by area below the reference axis.

Fundamental Theorem: The theorem relating a definite integral to an antiderivative.

Economic Surplus: The difference between benefits and market expenditure measured through integration.

Accumulation: The total quantity obtained by integrating a rate or density function.

16.9 EXERCISES

- 1 Evaluate the following definite integrals:
 - a) $\int_0^1 (3x^2 + 2x)dx$
 - b) $\int_2^4 (x^3 - 5x)dx$
- 2 Use properties of definite integrals to simplify the following expression:
 - a) $\int_0^5 (4x - 3)dx + \int_5^1 0(6x + 2)dx$
 - b) $\int_0^4 (2x^2 - 3x)dx - \int_4^6 (x^2 - 2x)dx$
- 3 Find the area under the curve for the following functions between the given limits:

- a) $y = x^2$ from $x = 0$ to $x = 3$
b) $y = \sin x$ from $x = 0$ to $x = \pi$
- 4 Find the area enclosed by the curve $y^2 = x$ and the straight line $x = 4$.
- 5 Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.
- 6 Find the area of a plate in the form of quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

16.10 SUGGESTED READINGS

1. T. M. Apostol, Calculus Vol I, Wiley & Sons (Asia) Pvt. Ltd. 1991.
2. Shanti Narayan, Integral Calculus, S. Chand, New Delhi, 2010.



BLOCK-V FINITE DIFFERENCES AND INTERPOLATION

UNIT 17 FINITE DIFFERENCES AND DIFFERENCE TABLE

STRUCTURE

- 17.1. Introduction
 - 17.2. Objectives
 - 17.3. Finite differences and operators
 - 17.4. Relation between operators
 - 17.5. Summary
 - 17.6. Glossary
 - 17.7. Exercises
 - 17.8. Suggested Readings
-

17.1 INTRODUCTION

In this unit, we delve into the concept of Finite Differences, an essential tool in numerical analysis used to approximate derivatives, solve differential equations, and analyze data trends. Finite differences provide a discrete counterpart to continuous derivatives, enabling computations in contexts where analytical solutions are challenging or unavailable.

We begin with the definition of finite differences, which captures the variation between successive terms in a sequence or function values over equal intervals. The most common types include forward, backward, and central differences.

Following the introduction of finite differences, we explore finite difference operators, which are mathematical symbols representing the finite difference calculations. The primary operators we study are:

Forward difference operator (Δ): Measures the change in function values by looking at successive terms.

Backward difference operator (∇): Measures changes by looking at previous terms.

Central difference operator (δ): Computes the difference based on the midpoint between successive terms.

We will also discuss the relation between these operators, focusing on their algebraic properties and how they can be transformed into one another. This understanding is crucial for building difference tables, which offer a systematic way to organize and compute differences for sequences or functions.

17.2 OBJECTIVES

After completing this unit, students will be able to

- Define and compute finite differences.
- Use finite difference operators effectively.
- Understand the relationship between different operators.

17.3 FINITE DIFFERENCES AND OPERATORS

Finite differences provide a way to approximate the derivative of a function by using discrete points. Given a function $y = f(x)$, the finite difference is defined as the difference between the function values at two points. Here, x is known as argument and y is called entry. Let the values of the argument are at equal interval.

$$a, a + h, a + 2h, a + 3h, \dots \dots \dots a + nh.$$

The corresponding values of y are

$$f(a), f(a + h), f(a + 2h), f(a + 3h), \dots \dots \dots f(a + nh).$$

The following differences are called finite difference

$$f(a + h) - f(a)$$

$$f(a + 2h) - f(a + h)$$

$$f(a + 3h) - f(a + 2h)$$

.....

.....

$$f(a + nh) - f(a + (n - 1)h)$$

The most common types of finite differences are:

- **Forward Difference:** $\Delta f(x) = f(x + h) - f(x)$
- **Backward Difference:** $\nabla f(x) = f(x) - f(x - h)$
- **Central Difference:** $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$

Forward Difference Operator: The forward difference operator Δ is defined as

$$\Delta f(x) = f(x + h) - f(x)$$

For $x = x_i$ we have

$$\Delta f(x_i) = f(x_i + h) - f(x_i)$$

$$\text{or} \quad \Delta y_i = y_{i+1} - y_i, \quad i = 0, 1, 2, \dots, n - 1$$

In particular, $\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2, \dots, \Delta y_{n-1} = y_n - y_{n-1}$. These are called first order differences.

Similarly, we can define second order differences by taking differences of first order. These differences are denoted by $\Delta^2 y_0, \Delta^2 y_1, \dots$ and defined as

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1.$$

$$\begin{aligned}\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) \\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

$$\Delta^3 y_1 = y_4 - 3y_3 + 3y_2 - y_1.$$

In general,

$$\Delta^{n+1} y_i = \Delta^n y_{i+1} - \Delta^n y_i, \quad i = 0, 1, 2, \dots$$

All the forward differences can be represented in a tabular form, called the forward difference or diagonal difference table.

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Properties of forward difference operator

- $\Delta[f_1(x) + f_2(x) + \dots + f_n(x)] = \Delta f_1(x) + \Delta f_2(x) + \dots + \Delta f_n(x)$
- $\Delta[c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)] = c_1 \Delta f_1(x) + c_2 \Delta f_2(x) + \dots + c_n \Delta f_n(x)$
- $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^n \Delta^m f(x) = \Delta^k \Delta^{m+n-k} f(x), k = 0, 1, 2, \dots, m \text{ or } n.$

Backward Difference Operator: The forward difference operator Δ is defined as

$$\nabla f(x) = f(x) - f(x + h)$$

For $x = x_i$ we have

$$\nabla f(x_i) = f(x_i) - f(x_i - h)$$

$$\text{or } \nabla y_i = y_i - y_{i-1}, \quad i = n(n-1), \dots, 1$$

In particular, $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$, $\nabla y_3 = y_3 - y_2, \dots, \nabla y_n = y_n - y_{n-1}$. These are called first order differences.

Similarly, we can define second order differences by taking differences of first order. These differences are denoted by $\nabla^2 y_2, \nabla^2 y_3, \dots, \nabla^2 y_n$. and defined as

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1.$$

In general,

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}, \quad i = n(n-1), \dots, 1$$

All the backward differences can be represented in a tabular form, called the backward difference or diagonal difference table.

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	$\nabla^2 y_2$		

$$x_3 \quad y_3 \quad \nabla y_3 \quad \nabla^2 y_3 \quad \nabla^3 y_3$$

$$x_4 \quad y_4 \quad \nabla y_4 \quad \nabla^2 y_4 \quad \nabla^3 y_4 \quad \nabla^4 y_4$$

Properties of forward difference operator

- $\nabla[f_1(x) + f_2(x) + \dots + f_n(x)] = \nabla f_1(x) + \nabla f_2(x) + \dots + \nabla f_n(x)$
- $\nabla[c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)] = c_1 \nabla f_1(x) + c_2 \nabla f_2(x) + \dots + c_n \nabla f_n(x)$
- $\nabla^m \nabla^n f(x) = \nabla^{m+n} f(x) = \nabla^n \nabla^m f(x) = \nabla^k \nabla^{m+n-k} f(x), k = 0, 1, 2, \dots, m \text{ or } n.$

Central Difference Operator: The central difference operator is denoted by δ and is defined by

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

For $x = x_i$ we have

$$\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$

Or

$$\delta y_i = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}$$

In particular, $\delta y_{\frac{1}{2}} = y_1 - y_0, \delta y_{\frac{3}{2}} = y_2 - y_1, \dots, \delta y_{n-\frac{1}{2}} = y_n - y_{n-1}.$

Similarly, we can define second order differences by taking differences of first order.

$$\delta^2 y_i = \delta y_{i+\frac{1}{2}} - \delta y_{i-\frac{1}{2}} = (y_{i+1} - y_i) - (y_i - y_{i-1}) = y_{i+1} - 2y_i + y_{i-1}.$$

In general,

$$\delta^n y_i = \delta^{n-1} y_{i+\frac{1}{2}} - \delta^{n-1} y_{i-\frac{1}{2}}.$$

All the central differences can be represented in a tabular form, called the central difference table.

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta y_{\frac{1}{2}}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{\frac{7}{2}}$			
x_4	y_4				

It may be observed that all odd (even) order differences have fraction suffices (integral suffices).

The Shift operator E : The shift operator is denoted by E and is defined by

$$Ef(x) = f(x + h)$$

Or $Ey_i = y_{i+1}$.

Note that shift operator increases subscript of y by one. When the shift operator is applied twice on the function $f(x)$, then the subscript of y is increased by 2.

$$E^2 f(x) = E[Ef(x)] = E[f(x + h)] = f(x + 2h).$$

In general,

$$E^n f(x) = f(x + nh) \text{ or } E^n y_i = y_i + nh.$$

The inverse shift operator can also find in similar manner. It is denoted by E^{-1} and is defined by

$$E^{-1} f(x) = f(x - h).$$

Similarly, second and higher order inverse operators are defined as follows:

$$E^{-2} f(x) = f(x - 2h) \text{ and } E^{-n} f(x) = f(x - nh).$$

Few common properties of E operator are given below:

(i) $Ec = c$, where c is a constant.

(ii) $E\{cf(x)\} = cEf(x)$.

(iii) $E\{c_1f_1(x) + c_2f_2(x) + \dots + c_nf_n(x)\} = c_1Ef_1(x) + c_2Ef_2(x) + \dots + c_nEf_n(x)$.

(iv) $E^m E^n f(x) = E^n E^m f(x) = E^{m+n} f(x)$.

(v) $E^n E^{-n} f(x) = f(x)$. In particular, $EE^{-1} \equiv I$, I is the identity operator and it is sometimes denoted by 1.

(vi) $(E^n)^m f(x) = E^{mn} f(x)$.

(vii) $E\{f(x)g(x)\} = Ef(x)Eg(x)$.

(viii) $E\left\{\frac{f(x)}{g(x)}\right\} = \frac{Ef(x)}{Eg(x)}$.

(ix) $E\Delta f(x) = \Delta Ef(x)$.

(x) $\Delta^m f(x) = \nabla^m E^m f(x) = E^m \nabla^m f(x)$ and $\nabla^m f(x) = \Delta^m E^{-m} f(x) = E^{-m} \Delta^m f(x)$.

The Average operator μ : The average operator is denoted by μ and is defined by

$$\mu f(x) = \frac{1}{2} \left[f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \right]$$

Or

$$\mu y_i = \frac{1}{2} \left[y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}} \right]$$

The Differential operator D : The differential operator is well known from differential calculus and it is denoted by D . This operator gives the derivative. That is,

$$Df(x) = \frac{d}{dx} f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f''(x)$$

.....

$$D^n f(x) = \frac{d^n}{dx^n} f(x) = f^n(x)$$

The factorial notation: The factorial notation is a very useful notation in calculus of finite difference. Using this notation one can find all order differences by the rules used in differential calculus. It is also very useful and simple notation to find anti-differences. The n th factorial of x is denoted by $x^{(n)}$ and is defined by

$$x^{(n)} = x(x - h)(x - 2h) \cdots (x - \overline{n-1}h),$$

where, each factor is decreased from the earlier by h ; and $x^{(0)} = 1$.

Similarly, the n th negative factorial of x is defined by

$$x^{(-n)} = \frac{1}{x(x + h)(x + 2h) \cdots (x + \overline{n-1}h)}.$$

A very interesting and obvious relation is $x^{(n)} \cdot x^{(-n)} = 1$. Following results show the similarity of factorial notation and differential operator.

Check Your Progress

1. For $y = x^2$ at $x = 1, 2, 3, 4, 5$, construct the forward difference table up to the second difference.

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2. Given y -values 10, 14, 20, 28, 38 for equally spaced x -values, find the first and second forward differences.

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3. For $y = x^3$ at $x = 0, 1, 2, 3, 4$, construct the difference table and identify the constant difference.

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17.4 RELATION BETWEEN OPERATORS

1. Relation between forward and backward difference operators:

We know $\Delta y_i = y_{i+1} - y_i = \nabla y_{i+1}$ and $\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i = \nabla^2 y_{i+2}$.

Therefore, in general we can write

$$\Delta^n y_i = \nabla^n y_{i+n}$$

2. Relation between forward difference and shift operators:

We know $\Delta f(x) = f(x+h) - f(x)$ and $Ef(x) = f(x+h)$, therefore

$$\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x) = (E-1)f(x)$$

So, we can write

$$\Delta = E - 1 \text{ or } E = 1 + \Delta$$

3. Relation between backward difference and shift operator:

We know $\nabla f(x) = f(x) - f(x-h)$ and $E^{-1}f(x) = f(x-h)$, therefore

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1}f(x) = (1 - E^{-1})f(x)$$

So, we can write

$$\nabla = 1 - E^{-1}$$

4. Relation between central difference and shift operator:

$$\begin{aligned} \delta f(x) &= f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) = E^{\frac{1}{2}}f(x) - E^{-\frac{1}{2}}f(x) \\ &= (E^{\frac{1}{2}} - E^{-\frac{1}{2}})f(x) \end{aligned}$$

So, we can write

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

5. Relation between average difference and shift operator:

We know

$$\begin{aligned} \mu f(x) &= \frac{1}{2} \left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right] = \frac{1}{2} \left[E^{\frac{1}{2}}f(x) + E^{-\frac{1}{2}}f(x) \right] = \frac{1}{2} (E^{\frac{1}{2}} + \\ &E^{-\frac{1}{2}})f(x) \end{aligned}$$

So, we can write

$$\mu = \frac{1}{2} [E^{\frac{1}{2}} + E^{-\frac{1}{2}}]$$

6. Relation between average difference and central operator:

By previous relation we know that $\mu f(x) = \frac{1}{2} (E^{\frac{1}{2}} + E^{-\frac{1}{2}})f(x)$, therefore

$$\begin{aligned} \mu^2 f(x) &= \frac{1}{4} \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right]^2 f(x) = \frac{1}{4} \left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right)^2 + 4 \right] f(x) \\ &= \frac{1}{4} [\delta^2 + 4]f(x). \end{aligned}$$

So, we can write

$$\mu = \sqrt{1 + \frac{1}{4}\delta^2}$$

7. Relation between forward, backward, central and shift operator:

We have $\nabla E f(x) = \nabla f(x + h) = f(x + h) - f(x) = \Delta f(x)$.

And $\delta E^{\frac{1}{2}} f(x) = \delta f\left(x + \frac{h}{2}\right) = f(x + h) - f(x) = \Delta f(x)$.

So, we can write

$$\Delta = \nabla E = \delta E^{1/2}$$

8. Relation between differential and shift operator:

We know

$$E f(x) = f(x + h)$$

Expand this by Taylor's series in the powers of h , we get

$$\begin{aligned} E f(x) &= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \\ &= f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \dots \\ &= \left[1 + hD + \frac{h^2}{2!} D^2 + \frac{h^3}{3!} D^3 + \dots \right] f(x) \\ &= e^{hD} f(x). \end{aligned}$$

So, we can write

$$E = e^{hD}$$

9. Relation between differential and central difference operator:

We know $\delta f(x) = [E^{\frac{1}{2}} - E^{-\frac{1}{2}}] f(x)$. Then by using previous relation $E = e^{hD}$ we get

$$\begin{aligned} \delta f(x) &= [E^{\frac{1}{2}} - E^{-\frac{1}{2}}] f(x) = [e^{\frac{hd}{2}} - e^{-\frac{hd}{2}}] f(x) \\ &= 2 \sinh\left(\frac{hD}{2}\right) f(x) \end{aligned}$$

So, we can write

$$\delta = 2 \sinh\left(\frac{hD}{2}\right)$$

9. Relation between differential and average difference operator:

We know $\mu f(x) = [E^{\frac{1}{2}} + E^{-\frac{1}{2}}]f(x)$. Then by using previous relation $E = e^{hD}$ we get

$$\begin{aligned}\delta f(x) &= \frac{1}{2}[E^{\frac{1}{2}} + E^{-\frac{1}{2}}]f(x) = \frac{1}{2}[e^{\frac{hd}{2}} + e^{-\frac{hd}{2}}]f(x) \\ &= \cosh\left(\frac{hD}{2}\right) f(x)\end{aligned}$$

So, we can write

$$\mu = \cosh\left(\frac{hD}{2}\right)$$

10. From the last two relation we have $\delta = 2 \sinh\left(\frac{hD}{2}\right)$ and $\mu = \cosh\left(\frac{hD}{2}\right)$.

By multiplying these two relations we have

$$\begin{aligned}\delta\mu &= 2 \sinh\left(\frac{hD}{2}\right) \cosh\left(\frac{hD}{2}\right) = \sinh(hD) \\ \Rightarrow hD &= \sinh^{-1}(\mu\delta)\end{aligned}$$

In the following table, it is shown that any operator can be expressed with the help of another operator

Table 17.1 Relation between difference operators

	E	Δ	∇	δ	hD
E	E	$\Delta + 1$	$(1 - \nabla)^{-1}$	$1 + \frac{\delta^2}{2} + \delta \sqrt{1 + \frac{1}{4}\delta^2}$	e^{hD}

Δ	$E - 1$	Δ	$(1 - \nabla)^{-1} - 1$	$\frac{\delta^2}{2} + \delta \sqrt{1 + \frac{1}{4}\delta^2}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1 + \Delta)^{-1}$	∇	$-\frac{\delta^2}{2} + \delta \sqrt{1 + \frac{1}{4}\delta^2}$	$1 - e^{hD}$
δ	$E^{\frac{1}{2}} - E^{-\frac{1}{2}}$	$\Delta(1 + \Delta)^{-1/2}$	$\Delta(1 - \Delta)^{-1/2}$	δ	$2 \sinh\left(\frac{hD}{2}\right)$
μ	$\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$	$\left(1 + \frac{\Delta}{2}\right) \times (1 + \Delta)^{-1/2}$	$\left(1 - \frac{\nabla}{2}\right) \times (1 - \nabla)^{-1/2}$	$\sqrt{1 + \frac{1}{4}\delta^2}$	$\cosh\left(\frac{hD}{2}\right)$
hD	$\log E$	$\log(1 + \Delta)$	$-\log(1 - \Delta)$	$2 \sinh^{-1}(\delta/2)$	hd

17.5 SUMMARY

In this unit, we explored the concept of finite differences and how they help approximate changes in discrete data or functions. We introduced key finite difference operators: the forward (Δ), backward (∇), and central (δ) difference operators, and learned how to compute and relate them. Additionally, we understood the importance of difference tables for organizing and calculating successive differences, which play a crucial role in interpolation and numerical solutions of differential equations. Mastery of these tools is essential for numerical analysis and various applications in computational mathematics.

17.6 GLOSSARY

Finite Difference: The difference between successive values of a function.

Forward Difference: A difference formed using a value and the following value.

Backward Difference: A difference formed using a value and the preceding value.

Central Difference: A difference based on values on both sides of a point.

Difference Table: A tabular arrangement of successive finite differences.

First Difference: The difference between consecutive function values.

Second Difference: The difference between consecutive first differences.

Equal Intervals: Data points separated by the same interval in the independent variable.

17.7 EXERCISES

1. Compute the forward differences for the following sequence:

5, 12, 26, 47, 75, 110. Construct the forward difference table up to the second order.

2. Find the backward differences for the sequence:

3, 7, 13, 21, 31, 43. Construct the backward difference table up to the second order.

3. Given the function $f(x) = x^2 + 3x + 2$, calculate the forward difference $\Delta f(x)$ for values of x from 0 to 5 with a step size of 1.

4. Find the relationship between the forward difference operator Δ and backward difference operator ∇ for the sequence: (2, 6, 12, 20, 30, 42).

5. Construct the central difference table for the following data set:

$x = 1, 2, 3, 4, 5$ and $f(x) = 2, 5, 10, 17, 26$.

6. Prove the relation $(\Delta = \nabla + E)$, where Δ is the forward difference operator, ∇ is the backward difference operator, and E is the shift operator.
7. Using finite differences, determine whether the function $f(x) = 3x^2 + 2x + 1$ is a polynomial of degree 2 by calculating the second-order differences.
8. Determine the third forward difference for the function values:
$$f(0) = 3, f(1) = 8, f(2) = 15, f(3) = 24, f(4) = 35.$$
9. If $\Delta f(x) = 5x + 3$, find the general expression for $f(x)$ using the properties of finite difference operators.
10. For the data set $x = 1, 2, 3, 4, 5$ and $f(x) = 1, 4, 9, 16, 25$, compute the forward, backward, and central differences. Compare the results and discuss the significance of each.

17.8 SUGGESTED READINGS

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UNIT 18 NEWTON'S FORWARD AND BACKWARD FORMULA

STRUCTURE

- 18.1. Introduction
 - 18.2. Objectives
 - 18.3. Newton's forward interpolation formula
 - 18.4. Newton's backward interpolation formula
 - 18.5. Summary
 - 18.6. Glossary
 - 18.7. Exercises
 - 18.8. Suggested Readings
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18.1 INTRODUCTION

Newton's Interpolation Formulae are fundamental in numerical methods for estimating functions using known data points. Developed by Isaac Newton, these formulae construct polynomial equations that pass through the given points, enabling the prediction of function values at intermediate locations. Newton's Forward Interpolation Formula uses forward differences to estimate values starting from the initial data point, while Newton's Backward Interpolation Formula applies backward differences, working from the last data point to approximate intermediate values. Both methods are essential for accurate and flexible function approximation across various numerical applications.

18.2 OBJECTIVES

After completing this unit, students will be able to

- Remember Newton's forward and backward interpolation formula
- Apply Newton's forward and backward interpolation formula to solve problems

18.3 NEWTON'S FORWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a function of x which assumes the values $f(a), f(a + h), f(a + 2h), \dots, f(a + nh)$ for $(n + 1)$ equidistant values $a, a + h, a + 2h, \dots, a + nh$ of the independent variable x . Let $f(x)$ be a polynomial of n^{th} degree.

$$\begin{aligned} \text{Let } f(x) = A_0 + A_1(x - a) + A_2(x - a)(x - a - h) \\ + A_3(x - a)(x - a - h)(x - a - 2h) + \dots \end{aligned} \quad (1)$$

where $A_0, A_1, A_2, \dots, A_n$ are to be determined.

Put $x = a, a + h, a + 2h, \dots, a + nh$ in (1) successively.

For $x = a, f(a) = A_0$

For $x = a + h, f(a + h) = A_0 + A_1h$

$$\Rightarrow f(a + h) = f(a) + A_1h$$

$$\Rightarrow A_1 = \frac{\Delta f(a)}{h}$$

For $x = a + 2h,$

$$f(a + 2h) = A_0 + A_1(2h) + A_2(2h)h$$

$$= f(a) + 2h \left\{ \frac{\Delta f(a)}{h} \right\} + 2h^2 A_2$$

$$\Rightarrow 2h^2 A_2 = f(a + 2h) - 2f(a + h) + f(a) = \Delta^2 f(a)$$

$$\Rightarrow A_2 = \frac{\Delta^2 f(a)}{2! h^2}$$

Similarly, $A_3 = \frac{\Delta^3 f(a)}{3! h^3}$ and so on. Thus, $A_n = \frac{\Delta^n f(a)}{n! h^n}$.

$$\text{From (1), } f(x) = f(a) + (x-a) \frac{\Delta f(a)}{h} + (x-a)(x-a-h) \frac{\Delta^2 f(a)}{2! h^2} + \dots \\ + (x-a) \dots (x-a-(n-1)h) \frac{\Delta^n f(a)}{n! h^n}$$

Put $x = a + hu \Rightarrow u = \frac{x-a}{h}$, we have

$$f(a+hu) = f(a) + hu \frac{\Delta f(a)}{h} + \frac{(hu)(hu-h)}{2! h^2} \Delta^2 f(a) + \dots \\ + \frac{(hu)(hu-h)(hu-2h) \dots (hu-(n-1)h)}{n! h^n} \Delta^n f(a)$$

$$f(a+hu) = f(a) + u \Delta f(a) + \frac{u(u-1)}{2!} \Delta^2 f(a) + \dots + \frac{u(u-1)(u-2) \dots (u-n+1)}{n!} \Delta^n f(a) \dots (2)$$

which is the required **Newton's forward interpolation** formula.

This formula is particularly useful for interpolating the values of $f(x)$ near the beginning of the set of values given h is called the interval of difference, while Δ is forward difference operator.

Example 1. The values of $f(x)$ for $x = 0, 1, 2, \dots, 6$ are given by

$x:$	0	1	2	3	4	5	6
$f(x):$	2	4	10	16	20	24	38

Estimate the value of $f(3.2)$ using only four of the given values. Choose the four values that you think will give the best approximation.

Solution. Last four values of $f(x)$ for $x = 3, 4, 5, 6$ are taken into consideration so that 3.2 occurs in the beginning of the table.

Here $a = 3, h = 1, x = 3.2 \Rightarrow a + hu = 3.2 \Rightarrow u = 0.2$

The difference table is:

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
3	16	4	0	10
4	20	4	10	
5	24	14		
6	38			

Applying Newton's forward difference formula,

$$\begin{aligned}
 f(3.2) &= f(3) + u\Delta f(3) + \frac{u(u-1)}{2!}\Delta^2 f(3) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(3) \\
 &= 16 + (.2)(4) + \frac{(.2)(.2-1)}{2}(0) + \frac{(.2)(.2-1)(.2-2)}{6}(10) = 17.28.
 \end{aligned}$$

Example 2. Find the value of $\sin 52^\circ$ from the given table:

θ°	45°	50°	55°	60°
$\sin \theta$	0.7071	0.7660	0.8192	0.8660

Solution.

$$a = 45^\circ, h = 5, x = 52$$

$$\therefore u = \frac{x-a}{h} = \frac{7}{5} = 1.4$$

Difference table is:

x°	Differences			
	$10^4 y$	$10^4 \Delta y$	$10^4 \Delta^2 y$	$10^4 \Delta^3 y$
45°	7071	589	-57	-7
50°	8660	532	-64	
55°	8192	468		
60°	8660			

By forward difference formula,

$$f(a + hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$\Rightarrow 10^4 f(x) = 10^4 f(a) + 10^4 u\Delta f(a) + 10^4 \frac{u(u-1)}{2!}\Delta^2 f(a) + 10^4 \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$\Rightarrow 10^4 f(52) = 10^4 f(45) + (1.4)10^4 \Delta f(45) + \frac{(1.4)(1.4-1)}{2!}10^4 \Delta^2 f(45) + \frac{(1.4)(1.4-1)(1.4-2)}{3!}10^4 \Delta^3 f(45)$$

$$= 7071 + (1.4)(589) + \frac{(1.4)(.4)}{2}(-57) + \frac{(1.4)(.4)(-.6)}{6}(-7)$$

$$= 7880$$

$$\therefore f(52) = .7880. \text{ Hence, } \sin 52^\circ = 0.7880.$$

Check Your Progress

1. Using Newton's forward interpolation formula, estimate y at $x = 1.5$ from the data: $x = 0, 1, 2, 3$; $y = 1, 3, 7, 13$.

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2. Using Newton's backward interpolation formula, estimate y at $x = 2.5$ from the data: $x = 0, 1, 2, 3$; $y = 2, 5, 10, 17$.

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3. For the data $x = 10, 20, 30, 40$ and $y = 25, 40, 55, 70$, use an appropriate Newton interpolation formula to estimate y at $x = 25$.

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18.4 NEWTON'S BACKWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a function of x which assumes the values $f(a), f(a + h), f(a + 2h), \dots, f(a + nh)$ for $(n + 1)$ equidistant values $a, a + h, a + 2h, \dots, a + nh$ of the independent variable x .

Let $f(x)$ be a polynomial of the n^{th} degree.

Let, $f(x) = A_0 + A_1(x - a - nh) + A_2(x - a - nh)(x - a - \overline{n - 1}h) + \dots$

$$+A_n(x-a-nh)(x-a-\overline{n-1}h) \dots (x-a-h) \quad (1)$$

where $A_0, A_1, A_2, A_3, \dots, A_n$ are to be determined.

Put $x = a + nh, a + \overline{n-1}h, \dots, a$ in (23) respectively.

Put $x = a + nh$, then $f(a + nh) = A_0$

Put $x = a + (n-1)h$, then

$$f(a + \overline{n-1}h) = A_0 - h A_1 = f(a + nh) - h A_1 \quad (2)$$

$$\Rightarrow A_1 = \frac{\nabla f(a + nh)}{h} \quad (3)$$

Put $x = a + (n-2)h$, then

$$f(a + \overline{n-2}h) = A_0 - 2h A_1 + (-2h)(-h)A_2$$

$$\Rightarrow 2! h^2 A_2 = f(a + \overline{n-2}h) - f(a + nh) + 2\nabla f(a + nh) \\ = \nabla^2 f(a + nh)$$

Proceeding, we get

$$A_n = \frac{\nabla^n f(a + nh)}{n! h^n} \quad (4)$$

Substituting the values in (24), we get

$$f(x) = f(a + nh) + (x - a - nh) \frac{\nabla f(a + nh)}{h} + \dots \\ + (x - a - nh)(x - a - \overline{n-1}h) \quad (5)$$

Put $x = a + nh + uh$, then

$$x - a - nh = uh$$

and

$$x - a - (n-1)h = (u+1)h$$

$\vdots$

$$x - a - h = (u + \overline{n-1})h$$

∴ (5) becomes,

$$f(x) = f(a + nh) + u\nabla f(a + nh) + \frac{u(u+1)}{2!} \nabla^2 f(a + nh) + \dots + uh(u+1)h \dots (u+n-1)h \frac{\nabla^n f(a + nh)}{n! h^n}$$

or

$$f(a + nh + uh) = f(a + nh) + u\nabla f(a + nh) + \frac{u(u+1)}{2!} \nabla^2 f(a + nh) + \dots + \frac{u(u+1) \dots (u+n-1)}{n!} \nabla^n f(a + nh)$$

which is the required formula.

This formula is useful when the value of $f(x)$ is required near the end of the table.

Example 1. The population of a town is as follows:

Year: 1921 1931 1941 1951 1961 1971

Population: 20 24 29 36 46 51 (in Lakhs)

Estimate the increase in population during the period 1955 to 1961.

Solution. Here , $a + nh = 1971, h = 10, a + nh + uh = 1955$

$$\therefore 1971 + 10u = 1955 \Rightarrow u = -1.6$$

The difference table is:

X	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1921	20					
1931	24	4				
1941	29	5	1			

1951	36	7	2	1		
1961	46	10	3	1	0	
1971	51	5	-5	-8	-9	-9

Applying Newton's backward difference formula, we get

$$\begin{aligned}
 y_{1955} &= y_{1971} + u\nabla y_{1971} + \frac{u(u+1)}{2!}\nabla^2 y_{1971} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{1971} \\
 &\quad + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{1971} + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_{1971} \\
 &= 51 + (-1.6)(5) + \frac{(-1.6)(-0.6)}{2!}(-5) + \frac{(-1.6)(-0.6)(0.4)}{6}(-8) \\
 &\quad + \frac{(-1.6)(-0.6)(.4)(1.4)}{24}(-9) + \frac{(-1.6)(-0.6)(0.4)(1.4)(2.4)}{120}(-9) \\
 &= 39.789632
 \end{aligned}$$

$\therefore$ Increase in population during period 1955 to 1961 is

$$\begin{aligned}
 &= 46 - 39.789632 = 6.210368 \text{ Lakhs} \\
 &= 621036.8 \text{ Lakhs.}
 \end{aligned}$$

Example 2. The population of a town was as given. Estimate the population for the year 1925.

Year (x):	1891	1901	1911	1921	1931
Population (in thousands) (y):	46	66	81	93	101

Solution. Here , $a + nh = 1931$, $h = 10$, $a + nh + uh = 1925$

$$\therefore u = \frac{1925 - 1931}{10} = -0.6$$

The difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
---	---	------------	--------------	--------------	--------------

189 1	46				
190 1	66	20			
191 1	81	15	-5		
192 1	93	12	-3	2	
193 1	101	8	-4	-1	-3

Applying Newton's Backward difference formula, we get

$$\begin{aligned}
 y_{1925} &= y_{1931} + u \nabla y_{1931} + \frac{u(u+1)}{2!} \nabla^2 y_{1931} \\
 &\quad + \frac{u(u+1)(u+2)}{3!} \nabla^3 y_{1931} + \frac{u(u+1)(u+2)(u+3)}{4!} \nabla^4 y_{1931} \\
 &= 101 + (-.6)(8) + \frac{(-.6)(.4)}{2!} (-4) + \frac{(-.6)(.4)(1.4)}{3!} (-1) \\
 &\quad + \frac{(-.6)(.4)(1.4)(2.4)}{4!} (-3) \\
 &= 96.8368 \text{ thousands.}
 \end{aligned}$$

Hence the population for the year 1925 = 96836.8 $\approx$ 96837.

Example 3. Given log x for x = 40, 45, 50, 55, 60 and 65 according to the following table:

x:	40	45	50	55	60	65
log x:	1.60206	1.65321	1.69897	1.74036	1.77815	1.81291

Find the value of log 5875.

Solution. The difference table is:

x	$10^5 \log x = 10^5 y_x$	$10^5 \nabla y_x$	$10^5 \nabla^2 y_x$	$10^5 \nabla^3 y_x$	$10^5 \nabla^4 y_x$	$10^5 \nabla^5 y_x$
40	160206					
45	165321	5115				
50	169897	4576	-539			
55	174036	4139	-437	102		
60	177815	3779	-360	77	-25	
65	181291	3476	-303	57	-20	5

Newton's Backward difference formula is

$$f(a + nh + uh) = f(a + nh) + u \nabla f(a + nh) + \frac{u(u+1)}{2!} \nabla^2 f(a + nh) + \frac{u(u+1)(u+2)}{3!} \nabla^3 f(a + nh) + \frac{u(u+1)(u+2)(u+3)}{4!} \nabla^4 f(a + nh) \quad (1)$$

First, we shall find the value of $\log(58.75)$.

Here, $a + nh = 65$, $h = 5$, $a + nh + uh = 58.75$

$$\therefore 65 + 5u = 58.75 \Rightarrow u = -1.25$$

From (1),

$$\begin{aligned} 10^5 f(58.75) &= 181291 + (-1.25)(3476) + \frac{(-1.25)(-.25)}{2!}(-303) \\ &+ \frac{(-1.25)(-.25)(.75)}{3!}(57) + \frac{(-1.25)(-.25)(.75)(1.75)}{4!}(-20) \\ &+ \frac{(-1.25)(-.25)(.75)(1.75)(2.75)}{5!}(5) \\ \Rightarrow 10^5 f(58.75) &= 176900.588 \\ \therefore f(58.75) &= \log 58.75 = 176900.588 \times 10^{-5} = 1.76900588 \end{aligned}$$

Hence,

$$\log 5875 = 3.76900588 \quad \therefore \text{Mantissa remain the same}$$

18.5 SUMMARY

In this unit, we have discussed the following concepts

- Newton's forward and backward interpolation formula
- Problems based on Newton's forward and backward interpolation formula

18.6 GLOSSARY

Interpolation: Estimating an unknown value within a given data range.

Newton's Forward Formula: An interpolation formula suited to values near the beginning of an equally spaced table.

Newton's Backward Formula: An interpolation formula suited to values near the end of an equally spaced table.

Forward Difference: The difference operator used in Newton's forward formula.

Backward Difference: The difference operator used in Newton's backward formula.

Step Size: The common interval between successive values of the independent variable.

Interpolation Point: The point at which an unknown value is estimated.

Equally Spaced Data: Data whose independent-variable values have a constant interval.

18.7 EXERCISES

1. The values of $f(x)$ for $x = 0, 1, 2, \dots, 6$ are given by

x:	0	1	2	3	4	5	6
f(x):	1	3	11	31	69	131	223

Estimate the value of $f(3.4)$, using only four of the given values.

2. The following table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x:	100	150	200	250	300	350	400
y:	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Use Newton's forward formula to find y when $x = 218$ ft.

3. If l_x represents the number of persons living at age x in a life table, find, as accurately as the data will permit, l_x for values of $x = 35, 42$ and 47 . Given

$$l_{20} = 512, l_{30} = 390, l_{40} = 360, l_{50} = 243.$$

4. The table below gives the value of $\tan x$ for $0.10 \leq x \leq 0.30$.

x:	0.10	0.15	0.20	0.25	0.30
$y = \tan x$:	0.1003	0.1511	0.2027	0.2553	0.3093

Find: (i) $\tan 0.50$ (ii) $\tan 0.26$ (iii) $\tan 0.40$.

5. Given:

x:	1	2	3	4	5	6	7	8
f(x):	1	8	27	64	125	216	343	512

Find $f(7.5)$ using Newton's Backward difference formula.

6. The area A of a circle of diameter d is given for the following values:

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105.

7. Estimate the value of $f(42)$ from the following available data:

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

18.8 SUGGESTED READINGS

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UNIT 19 GAUSS FORWARD AND BACKWARD INTERPOLATION FORMULA, STIRLING'S FORMULA

STRUCTURE

- 19.1. Introduction
- 19.2. Objectives
- 19.3. Gauss forward difference formula
- 19.4. Gauss backward difference formula
- 19.5. Stirling's formula
- 19.6. Application of interpolation in Economics
- 19.7. Summary
- 19.8. Glossary
- 19.9. Exercises
- 19.10. Suggested Readings

19.1 INTRODUCTION

In numerical analysis, interpolation is vital for estimating unknown function values based on a set of known data points. When the data points are evenly spaced, specialized interpolation formulas streamline the process, enhancing both accuracy and efficiency. This unit covers three key interpolation methods: Gauss Forward Interpolation, Gauss Backward Interpolation, and Stirling's Central Difference Formula.

These techniques are especially effective for evenly spaced data and are commonly used for estimating values near the center or edges of a dataset. Each method is tailored to specific situations, providing a reliable approach to function estimation within equal intervals.

1. Gauss Forward Interpolation Formula:

- Focused on interpolating values near the start of the data table.
- Introduces the concept of central differences, simplifying interpolation for equally spaced data points.
- Provides a step-by-step guide to constructing the Gauss Forward Interpolation formula.
- Applied to predict values when the point of interest is closer to the beginning of the dataset.

2. Gauss Backward Interpolation Formula:

- Applied for interpolation near the end of the data table.
- Highlights similarities with Gauss Forward Interpolation, but focuses on the dataset's endpoint.
- Explains the construction and use of the backward difference operator.
- Useful for estimating values near the final data points.

3. Stirling's Central Difference Formula:

- A symmetric interpolation method ideal for points near the center of the dataset.
- Merges forward and backward differences to develop an effective central interpolation technique.
- Derives Stirling's formula from Gauss forward and backward methods, highlighting its balanced nature for central points.

- Evaluates accuracy, particularly for data points that are equidistant from the dataset's midpoint.

19.2 OBJECTIVES

After completing this unit, students will be able to

- Remember and understand the Gauss forward and backward interpolation formula, Stirling's formula
- Apply the Gauss forward and backward interpolation formula, Stirling's formula

19.3 GAUSS' FORWARD DIFFERENCE FORMULA

Newton's forward difference formula is

$$f(a + hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \dots \dots \quad (1)$$

Given $a = 0$, $h = 1$, we get

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(0) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(0) + \dots \dots \quad (2)$$

Now, $\Delta^3 f(-1) = \Delta^2 f(0) - \Delta^2 f(-1) \Rightarrow \Delta^2 f(0) = \Delta^3 f(-1) + \Delta^2 f(-1)$

Also, $\Delta^4 f(-1) = \Delta^3 f(0) - \Delta^3 f(-1) \Rightarrow \Delta^3 f(0) = \Delta^4 f(-1) + \Delta^3 f(-1)$

and

$$\begin{aligned}\Delta^5 f(-1) &= \Delta^4 f(0) - \Delta^4 f(-1) \Rightarrow \Delta^4 f(0) \\ &= \Delta^5 f(-1) + \Delta^4 f(-1) \text{ and so on.}\end{aligned}$$

∴ From (2),

$$\begin{aligned}f(u) &= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!} \{\Delta^2 f(-1) + \Delta^3 f(-1)\} \\ &\quad + \frac{u(u-1)(u-2)}{3!} \{\Delta^3 f(-1) + \Delta^4 f(-1)\} \\ &\quad + \frac{u(u-1)(u-2)(u-3)}{4!} \{\Delta^4 f(-1) + \Delta^5 f(-1)\} + \dots \dots \\ &= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!} \Delta^2 f(-1) + \frac{u(u-1)}{2} \left\{1 + \frac{u-2}{3}\right\} \Delta^3 f(-1) \\ &\quad + \frac{u(u-1)(u-2)}{6} \left\{1 + \frac{u-3}{4}\right\} \Delta^4 f(-1) + \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^5 f(-1) + \dots \dots \\ &= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!} \Delta^3 f(-1)\end{aligned}$$

But, $\Delta^5 f(-2) = \Delta^4 f(-1) - \Delta^4 f(-2)$

$$\therefore \Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2)$$

therefore

$$\begin{aligned}f(u) &= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!} \Delta^3 f(-1) \\ &\quad + \frac{(u+1)u(u-1)(u-2)}{4!} \{\Delta^4 f(-2) + \Delta^5 f(-2)\} \\ &\quad + \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^5 f(-1) + \dots \dots\end{aligned}$$

$$\begin{aligned}f(u) &= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!} \Delta^3 f(-1) \\ &\quad + \frac{(u+1)u(u-1)(u-2)}{4!} \Delta^4 f(-2) + \dots \dots\end{aligned}$$

This is called Gauss' forward difference formula.

Example 1. Apply a central difference formula to obtain $f(32)$ given that:

$$\begin{aligned} f(25) &= 0.2707 & f(35) &= 0.3386 \\ f(30) &= 0.3027 & f(40) &= 0.3794. \end{aligned}$$

Solution. Here

$$a + hu = 32 \text{ and } h = 5$$

Take origin at 30 $\therefore a = 30$ then $u = 0.4$

The forward difference table is:

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	25	.2707	.032	.0039	.0010
0	30	.3027	.0359	.0049	
1	35	.3386	.0408		
2	40	.3794			

Applying Gauss' forward difference formula, we have

$$\begin{aligned} f(u) &= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) \\ \therefore f(.4) &= .3027 + (.4)(.0359) + \frac{(.4)(.4-1)}{2!}(.0039) + \frac{(1.4)(.4)(.4-1)}{3!}(.0010) \\ &= 0.316536. \end{aligned}$$

Example 2. Use Gauss' forward formula to find a polynomial of degree four which takes the following values of the function (x) :

$x:$ 1 2 3 4 5

$f(x):$ 1 -1 1 -1 1

Solution. Taking origin at 3 and $h = 1$

$$a + hu = x$$

$$\Rightarrow 3 + u = x \Rightarrow u = x - 3$$

The difference table is:

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-2	1	1	-2	4	-8	16
-1	2	-1	2	-4	8	
0	3	1	-2	4		
1	4	-1	2			
2	5	1				

Gauss' forward difference formula is

$$\begin{aligned}
f(u) &= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) \\
&+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2) \\
&= 1 + (x-3)(-2) + \frac{(x-3)(x-4)}{2}(-4) + \frac{(x-2)(x-3)(x-4)}{6}(8) \\
&+ \frac{(x-2)(x-3)(x-4)(x-5)}{24}(16) \\
&= 1 - 2x + 6 - 2x^2 + 14x - 24 + \frac{4}{3}(x^3 - 9x^2 + 26x - 24) \\
&+ \frac{2}{3}(x^4 - 14x^3 + 71x^2 - 154x + 120) \\
\therefore F(x) &= \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31
\end{aligned}$$

Example 3. Apply Gauss's forward formula to find the value of u_9 , if $u_0 = 14, u_4 = 24, u_8 = 32, u_{12} = 35, u_{16} = 40$.

Solution. The difference table is (taking origin at 8):

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(u)$	$\Delta^4 f(x)$
-2	0	14	10	-2	-3	10
-1	4	24	8	-5	7	
0	8	32	3	2		
1	12	35	5			
2	16	40				

Here

$$a = 8, h = 4, a + hu = 9$$

$$\therefore 8 + 4u = 9 \Rightarrow u = .25$$

Gauss' forward difference formula is

$$\begin{aligned} f(.25) &= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) \\ &\quad + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2) \\ &= 32 + (.25)(3) + \frac{(.25)(-.75)}{2}(-5) + \frac{(1.25)(.25)(-.75)}{6}(7) \\ &\quad + \frac{(1.25)(.25)(-.75)(-1.75)}{24}(10) \\ &= 33.11621094 \end{aligned}$$

Hence $u_9 = 33.11621094$.

Check Your Progress

1. Using Gauss' forward interpolation formula, estimate y at $x = 2.5$ from the equally spaced data $x = 0, 1, 2, 3, 4$ and $y = 1, 2, 5, 10, 17$.

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2. Using Gauss' backward interpolation formula, estimate y at $x = 1.5$ from the data $x = 0, 1, 2, 3, 4$ and $y = 3, 5, 9, 15, 23$.

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3. Using Stirling's interpolation formula, estimate y at x = 2.1 from the data x = 0, 1, 2, 3, 4 and y = 1, 2, 5, 10, 17.

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19.4 GAUSS'S BACKWARD DIFFERENCE FORMULA

Newton's forward difference formula is

$$f(a + hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \dots \dots (1)$$

Put $a = 0$, $h = 1$, we get

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(0) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(0) + \dots \dots (2)$$

Now, $\Delta f(0) = \Delta f(-1) + \Delta^2 f(-1)$

$$\Delta^2 f(0) = \Delta^2 f(-1) + \Delta^3 f(-1)$$

$$\Delta^3 f(0) = \Delta^3 f(-1) + \Delta^4 f(-1)$$

$$\Delta^4 f(0) = \Delta^4 f(-1) + \Delta^5 f(-1) \text{ and so on.}$$

$\therefore$ From (2),

$$\begin{aligned}
f(u) &= f(0) + u[\Delta f(-1) + \Delta^2 f(-1)] + \frac{u(u-1)}{2!} [\Delta^2 f(-1) + \Delta^3 f(-1)] \\
&+ \frac{u(u-1)(u-2)}{3!} [\Delta^3 f(-1) + \Delta^4 f(-1)] \\
&= f(0) + u\Delta f(-1) + u\left(1 + \frac{u-1}{2}\right) \Delta^2 f(-1) \\
&+ \frac{u(u-1)(u-2)}{6} \left\{1 + \frac{u-3}{4}\right\} \Delta^4 f(-1) + \frac{u(u-1)(u-2)(u-3)}{4!} \left(1 + \frac{u-2}{3}\right) \Delta^3 f(-1) \\
&= f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!} \Delta^3 f(-1) \\
&+ \frac{(u+1)u(u-1)(u-2)}{4!} \Delta^4 f(-1) + \dots \dots \dots (3)
\end{aligned}$$

Again, $\Delta^3 f(-1) = \Delta^3 f(-2) + \Delta^4 f(-2)$

and

$$\Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2) \text{ and so on}$$

$\therefore$ (3) gives

$$\begin{aligned}
f(u) &= f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!} \{\Delta^3 f(-2) \\
&\quad + \Delta^4 f(-2)\} \\
&\quad + \frac{(u+1)u(u-1)(u-2)}{4!} \{\Delta^4 f(-2) + \Delta^5 f(-2)\} + \dots \dots \\
f(u) &= f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!} \Delta^3 f(-2) + \dots (4)
\end{aligned}$$

This is known as Gauss' backward difference formula.

This formula is useful when u lies between $-\frac{1}{2}$ and 0.

Example 1. Given that

$$\begin{aligned}
\sqrt{12500} &= 111.803399, \sqrt{12510} = 111.848111 \\
\sqrt{12520} &= 111.892806, \sqrt{12530} = 111.937483
\end{aligned}$$

Show by Gauss's backward formula that $\sqrt{12516} = 111.8749301$.

Solution. Taking the origin at 12520

$$\therefore u = \frac{x - a}{h} = \frac{12516 - 12520}{10} = -\frac{4}{10} = -0.4$$

Gauss's backward formula is

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) \quad (1)$$

The difference table is:

u	x	$10^6 f(x)$	$10^6 \Delta f(x)$	$10^6 \Delta^2 f(x)$	$10^6 \Delta^3 f(x)$
-2	12500	111803399			
-1	12510	111848111	44712		
0	12520	111892806	44695	-17	
1	12530	111937483	44677	-18	-1

From (1),

$$\begin{aligned} 10^6 f(-.4) &= 111892806 + (-.4)(44695) \\ &= 111874930.1 \\ &\quad + \frac{(.6)(-.4)}{2!}(-18) + \frac{(.6)(-.4)(-1.4)}{3!}(-1) \end{aligned}$$

$$\text{Hence, } \sqrt{12516} = 111.8749301.$$

Example 2. Find the value of $\cos 51^\circ 42'$ by Gauss's backward formula.

Given that

$$x: \quad 50^\circ \quad 51^\circ \quad 52^\circ \quad 53^\circ \quad 54^\circ$$

$$\cos x: \quad 0.6428 \quad 0.6293 \quad 0.6157 \quad 0.6018 \quad 0.5878.$$

Solution. Taking the origin at 52° and $h = 1$

$$\therefore u = (x - a) = 51^\circ 42' - 52^\circ = -18' = -0.3^\circ$$

Gauss's backward formula is

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) \quad (1)$$

The difference table is as below:

u	x	$10^4 f(x)$	$10^4 \Delta f(x)$	$10^4 \Delta^2 f(x)$	$10^4 \Delta^3 f(x)$	$10^4 \Delta^4 f(x)$
-2	50°	6428				
-1	51°	6293	-135			
0	52°	6157	-136	-1		
1	53°	6018	-139	-3	2	
2	54°	5878	-140	-1		4

From (1),

$$\begin{aligned} 10^4 f(-.3) &= 6157 + (-.3)(-136) + \frac{(.7)(-.3)}{2!}(-3) + \frac{(.7)(-.3)(-1.3)}{3!}(-2) \\ &\quad + \frac{(1.7)(.7)(-.3)(-1.3)}{4!}(4) \\ &= 6198.10135 \\ \therefore f(-.3) &= .619810135 \end{aligned}$$

Hence $\cos 51^\circ 42' = 0.619810135$.

19.5 STIRLING'S FORMULA

Gauss's forward formula is

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \dots \quad (1)$$

Gauss's backward formula is

$$\begin{aligned} f(u) &= f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) \\ &\quad + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) + \dots \quad (2) \end{aligned}$$

Take the mean of (1) and (2),

$$f(u) = f(0) + u \left\{ \frac{\Delta f(0) + \Delta f(-1)}{2} \right\} + \frac{u^2}{2!} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!} \left\{ \frac{\Delta^3 f(-1) + \Delta^3 f(-2)}{2} \right\}$$

This is called Stirling's formula. It is useful when $|u| < \frac{1}{2}$ or $-\frac{1}{2} < u < \frac{1}{2}$. It gives the best estimate when $-\frac{1}{4} < u < \frac{1}{4}$.

Example 1. Given:

$\theta:$	0°	5°	10°	15°	20°	25°	30°
$\tan \theta:$	0	0.0875	0.1763	0.2672	0.3640	0.4663	0.5774

Find the value of $\tan 16^\circ$ using Stirling formula.

Solution. Take origin at 15°

$$\begin{aligned} \therefore a &= 15^\circ, h = 5 \\ a + hu &= 16 \\ \Rightarrow 15 + 5u &= 16 \Rightarrow u = .2 \end{aligned}$$

The difference table is:

u	θ	$10^4 f(\theta)$	$10^4 \Delta f(\theta)$	$10^4 \Delta^2 f(\theta)$	$10^4 \Delta^3 f(\theta)$	$10^4 \Delta^4 f(\theta)$	$10^4 \Delta^5 f(\theta)$	$10^4 \Delta^6 f(\theta)$
-3	0	0						
-2	5	875	875	13				
-1	10	1763	888	28	15			
0	15	2679	916	45	17	2	-2	
1	20	3640	961	62	17	9	9	11
2	25	4663	1023	88	26			
3	30	5774	1111					

Using Stirling's formula,

$$\begin{aligned}
10^4 f(.2) &= 2679 + (.2) \left(\frac{961 + 916}{2} \right) + \frac{(.2)^2}{2!} (45) + \frac{(1.2)(.2)(-.8)}{3!} \left(\frac{17 + 17}{2} \right) \\
&+ \frac{(.2)^2 \{ (.2)^2 - 1 \}}{4!} (0) + \frac{(2.2)(1.2)(.2)(-.8)(-1.8)}{5!} \left\{ \frac{9 + (-2)}{2} \right\} \\
&+ \frac{(.2)^2 \{ (.2)^2 - 1 \} \{ (.2)^2 - 4 \}}{6!} (11) \\
&= 2866.980499 \\
\therefore f(.2) &= .2866980499
\end{aligned}$$

Hence $\tan 16^\circ = 0.2866980499$.

Example 2. Apply Stirling's formula to find the value of $f(1.22)$ from the following table which gives the values of $f(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{x^2}{2}} dx$, at intervals of $x = 0.5$ from $x = 0$ to 2 .

x :	0	0.5	1.0	1.5	2.0
$f(x)$:	0	0.191	0.341	0.433	0.477.

Solution. Let the origin be at 1 and $h = 0.5$

$$\therefore x = a + hu, u = \frac{x - a}{h} = \frac{1.22 - 1.00}{0.5} = 0.44$$

Applying Stirling's formula

$$\begin{aligned}
f(u) &= f(0) + u \cdot \frac{1}{2} [\Delta f(0) + \Delta f(-1)] + \frac{u^2}{2!} \Delta^2 f(-1) \\
&+ \frac{u(u^2 - 1)}{3!} \cdot \frac{1}{2} [\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{u^2(u^2 - 1)}{4!} \cdot \Delta^4 f(-2) + \dots \dots \\
\therefore f(0.44) &= f(0) + (0.44) \frac{1}{2} [\Delta f(0) + \Delta f(-1)] + \frac{(0.44)^2}{2} \Delta^2 f(-1) \\
&+ \frac{(0.44)[(0.44)^2 - 1]}{6} \cdot \frac{1}{2} [\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{(0.44)^2[(0.44)^2 - 1]}{24} \Delta^4 f(-2) \\
&\simeq f(0) + (0.22)[\Delta f(0) + \Delta f(-1)] + 0.0968 \Delta^2 f(-1) \\
&- 0.029568 [\Delta^3 f(-1) + \Delta^3 f(-2)] - 0.06505 \Delta^4 f(-2) + \dots \dots
\end{aligned}$$

The difference table is as follows:

u	x	$10^3 f(x)$	$10^3 \Delta f(x)$	$10^3 \Delta^2 f(x)$	$10^3 \Delta^3 f(x)$	$10^3 \Delta^4 f(x)$
-2	0	0				
-1	.5	191	191			
0	1	341	150	-41	-17	
1	1.5	433	...	-58		
2	2	477	92	-48	10	27
			44			

$f(0)$ and the differences are being multiplied by 10^3

$$\begin{aligned}
 \therefore 10^3 f(0.44) &\simeq 341 + 0.22 \times (150 + 92) + 0.0968 \times (-58) \\
 &\quad - 0.029568 \times [-17 + 10] - 0.006505 \times 27 \\
 &\simeq 341 + 0.22 \times 242 - 0.0968 \times 58 + 0.029568 \times 7 - 0.006505 \times 27 \\
 &\simeq 341 + 53.24 - 5.6144 + 0.206276 - 0.175635 \simeq 388.66 \\
 \therefore \\
 \square f(0.44) &= 0.389
 \end{aligned}$$

Hence the required value of $f(x)$ at $x = 1.22$ is 0.389.

19.6 APPLICATIONS OF INTERPOLATION IN ECONOMICS

1. Estimating Missing Data:

- Interpolation helps estimate economic indicators (e.g., GDP, inflation, or unemployment rates) for periods where data is unavailable or incomplete, ensuring smoother analysis over time.

2. Trend Analysis and Forecasting:

- By interpolating between known data points, economists can identify trends and predict future economic performance, aiding in policy formulation and market analysis.

3. Price and Demand Estimation:

- Interpolation is used to estimate consumer demand or price levels at points not directly measured, supporting demand-supply modeling and elasticity calculations.
- 4. National Income and Productivity Measurement:**
 - For sectors with incomplete records, interpolation helps approximate national income or productivity levels, contributing to more accurate economic growth assessments.
- 5. Index Construction:**
 - Economic indices (like cost of living or price indices) often require interpolating data points to build comprehensive datasets, reflecting changes over time more accurately.
- 6. Market Research and Analysis:**
 - In market research, interpolation estimates market size, revenue, or consumer preferences at different price levels or regions where direct data collection isn't feasible.
- 7. Resource Allocation:**
 - Governments and organizations use interpolation to estimate the impact of policy changes or resource distribution across different regions, filling gaps in socioeconomic data.
- 8. Wage and Income Distribution:**
 - Interpolation can estimate wage growth or income distribution trends, helping economists analyze disparities and propose targeted economic policies.

Interpolation is a critical tool in economic modeling, ensuring more accurate predictions and comprehensive analysis, even when complete datasets are unavailable.

19.7 SUMMARY

In this unit, we have discussed the following concepts

- Gauss forward and backward interpolation formula,

- Stirling's formula
- Problems based on Gauss forward and backward interpolation
- Problems based on Stirling's formula

19.8 GLOSSARY

Gauss' Forward Formula: A central interpolation formula used near the middle of an equally spaced table.

Gauss' Backward Formula: A central interpolation formula based on backward differences.

Stirling's Formula: A central interpolation formula using symmetric differences.

Central Difference: A difference formed symmetrically around a central value.

Central Value: The selected middle value used as the reference point.

Interpolation Parameter: A normalized quantity measuring distance from the central point.

Symmetric Difference: A difference constructed using values on both sides of a point.

Central Interpolation: Interpolation performed using data surrounding a central point.

19.9 EXERCISES

1. From the following table, find the value of $e^{1.17}$ using Gauss forward formula:

$x:$	1	1.05	1.10	1.15	1.20	1.25	1.30
$e^x:$	2.718	2.857	3.004	3.158	3.320	3.490	3.669
	3	7	2	2	1	3	3

2. Find the value of $f(41)$ by applying Gauss's forward formula from the following data:

$x:$ 30 35 40 45 50

$f(x):$ 3678.2 2995.1 2400.1 1876.2 1416.3

3. Given that

$x:$ 25 30 35 40 45

$\log x:$ 1.39794 1.47712 1.54407 1.60206 1.65321

Find the value of $\log 3.7$, using Gauss's forward formula.

4. Apply Gauss's backward formula to find $\sin 45^\circ$ from the following table:

$\theta^\circ:$	20	30	40	50	60	70	80
$\sin \theta:$	0.34202	0.50002	0.64279	0.76604	0.86603	0.93969	0.98481

5. Interpolate by means of Gauss's backward formula, the population of a town for the year 1974 given that:

Year: 1939 1949 1959 1969 1979 1989

Population:

(in thousands) 12 15 20 27 39 52

6. Apply Gauss's backward formula to find the value of $(1.06)^{19}$ if $(1.06)^{10} = 1.79085$, $(1.06)^{15} = 2.39656$, $(1.06)^{20} = 3.20714$, $(1.06)^{25} = 4.29187$ and $(1.06)^{30} = 5.74349$.

7. Use Stirling's formula to find u_{32} from the following table:

$u_{20} = 14.035$	$u_{30} = 13.257$
$u_{40} = 12.089$	$u_{25} = 13.674$
$u_{35} = 12.734$	$u_{45} = 11.309$

8. Find $f(0.41)$ using Stirling's formula, if

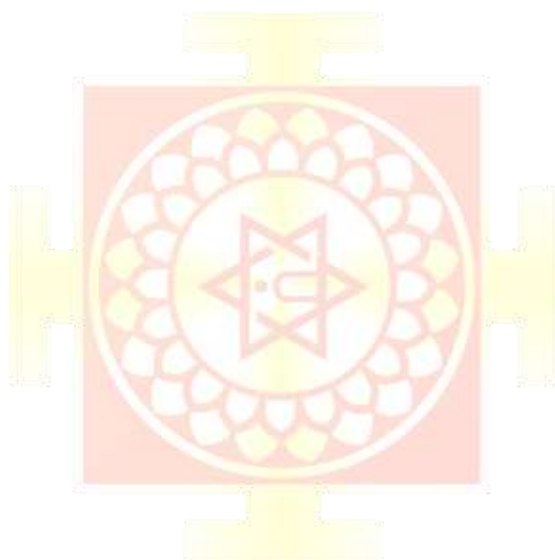
$f(0.30) = 0.1179$, $f(0.35) = 0.1368$, $f(0.40) = 0.1554$
 $f(0.45) = 0.1736$, $f(0.50) = 0.1915$

9. Employ Stirling's formula to evaluate $y_{12.2}$ from the following table ($y_x = 1 + \log_{10} \sin x$) :

x° :	10	11	12	13	14
$10^5 y_x$:	23967	28060	31788	35209	38368.

19.10 SUGGESTED READINGS

1. Ranganatham S., Prasad M. V. S. S. N. and Ramesh Babu V.; Numerical Analysis, S. Chand Publishing, 2022.
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UNIT 20 INTERPOLATION WITH UNEQUAL INTERVALS

STRUCTURE

- 20.1. Introduction
- 20.2. Objectives
- 20.3. Lagrange's interpolation formula
- 20.4. Examples
- 20.5. Summary
- 20.6. Glossary
- 20.7. Exercises
- 20.8. Suggested Readings

20.1 INTRODUCTION

Interpolation is a key technique in numerical analysis for estimating unknown function values from known data points. In real-world applications, data points are often unevenly spaced, making interpolation over unequal intervals necessary. Lagrange's Interpolation Formula is a highly effective method for handling such cases.

This unit explores the principles and applications of Lagrange's interpolation formula to approximate functions using irregularly spaced data points. Unlike methods that rely on equal spacing, Lagrange's formula offers flexibility, is straightforward to apply, and provides accurate results even when data intervals vary.

1. The Need for Unequal Interval Interpolation:

Importance of handling data points that are not evenly distributed.

Applications in fields such as engineering, physics, and data science, where measurements are often irregular.

2. Lagrange's Interpolation Formula:

Detailed derivation of Lagrange's formula for constructing the interpolating polynomial.

Understanding how Lagrange polynomials are used to estimate the value of a function at any given point within the range of the known data.

3. Advantages of Lagrange's Formula:

Flexibility in handling non-uniform data.

No need to calculate divided differences, unlike Newton's method.

Simple structure, with explicit terms that involve only the given data points.

By the end of this unit, you will be able to use Lagrange's interpolation formula to interpolate functions from irregularly spaced data points, understand its advantages and limitations, and apply it to solve practical problems. This knowledge is crucial for fields where data is not uniformly distributed and precise function approximation is needed.

20.2 OBJECTIVES

After completing this unit, students will be able to

- Remember and understand the Lagrange's interpolation formula for unequal intervals
- Apply Lagrange's interpolation formula for unequal intervals to solve problems

20.3 LAGRANGE'S INTERPOLATION FORMULA

Let $f(x_0), f(x_1), \dots, f(x_n)$ be $(n + 1)$ entries of a function $y = f(x)$, where $f(x)$ is assumed to be a polynomial corresponding to the arguments $x_0, x_1, x_2, \dots, x_n$.

The polynomial $f(x)$ may be written as

$$f(x) = A_0(x - x_1)(x - x_2) \dots (x - x_n) + A_1(x - x_0)(x - x_2) \dots (x - x_n) + \dots (1)$$

where $A_0, A_1, \dots, A_n$ are constants to be determined.

Putting $x = x_0, x_1, \dots, x_n$ in (1), we get

$$\begin{aligned} f(x_0) &= A_0(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n) \\ \therefore A_0 &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} \\ f(x_1) &= A_1(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n) \\ \therefore A_1 &= \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} \end{aligned} \quad (2)$$

Similarly,

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} \quad (3)$$

Substituting the values of $A_0, A_1, \dots, A_n$ in equation (1), we get

$$\begin{aligned} f(x) &= \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} f(x_0) \\ &+ \frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} f(x_1) \\ &+ \dots + \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} f(x_n) \end{aligned} \quad (4)$$

This is called Lagrange's Interpolation Formula. In eqn. (4), dividing both sides by $(x - x_0)(x - x_1) \dots (x - x_n)$, Lagrange's formula may also be written

$$\begin{aligned} \frac{f(x)}{(x - x_0)(x - x_1) \dots (x - x_n)} &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} \cdot \frac{1}{(x - x_0)} \\ &+ \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} \cdot \frac{1}{(x - x_1)} + \dots \\ &+ \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} \cdot \frac{1}{(x - x_n)}. \end{aligned} \quad (5)$$

Another form of Lagrange's Formula

Prove that the Lagrange's formula can be put in the form

$$P_n(x) = \sum_{r=0}^n \frac{\phi(x)f(x_r)}{(x-x_r)\phi'(x_r)}$$

where

$$\phi(x) = \prod_{r=0}^n (x-x_r)$$

and

$$\phi'(x_r) = \left[\frac{d}{dx} \{\phi(x)\} \right]_{x=x_r}$$

We have the Lagrange's formula,

$$P_n(x) = \sum_{r=0}^n \frac{(x-x_0)(x-x_1) \dots (x-x_{r-1})(x-x_{r+1}) \dots (x-x_n)}{(x_r-x_0)(x_r-x_1) \dots (x_r-x_{r-1})(x_r-x_{r+1}) \dots (x_r-x_n)} f(x_r) \quad (6)$$

Now,

$$\begin{aligned} \phi(x) &= \prod_{r=0}^n (x-x_r) \\ &= (x-x_0)(x-x_1) \dots (x-x_{r-1})(x-x_r)(x-x_{r+1}) \dots (x-x_n) \\ \therefore \phi'(x) &= (x-x_1)(x-x_2) \dots (x-x_r) \dots (x-x_n) \\ &\quad + (x-x_0)(x-x_2) \dots (x-x_r) \dots (x-x_n) + \dots \\ &\quad + (x-x_0)(x-x_1) \dots (x-x_{r-1})(x-x_{r+1}) \dots (x-x_n) + \dots \\ &\quad + (x-x_0)(x-x_1) \dots (x-x_r) \dots (x-x_{n-1}) \\ &\Rightarrow \phi'(x_r) = [\phi'(x)]_{x=x_r} \end{aligned}$$

Hence from (6),

$$P_n(x) = \sum_{r=0}^n \frac{\phi(x)f(x_r)}{(x - x_r)\phi'(x_r)} \quad (7)$$

Check Your Progress

1. Using Lagrange's interpolation formula, estimate y at x = 2 from the data (x, y) = (1, 2), (3, 10), (4, 17).

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2. Using Lagrange's formula, estimate y at x = 2.5 from the data (1, 3), (2, 6), (4, 18).

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3. Using Lagrange's interpolation formula for unequal intervals, estimate y at x = 5 from the data (2, 4), (3, 9), (7, 49).

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20.4 EXAMPLES

Example 1. Using Lagrange's interpolation formula, find $y(10)$ from the following table:

x	5	6	9	11
y	12	13	14	16

Solution. Here $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$
 $f(x_0) = 12$, $f(x_1) = 13$, $f(x_2) = 14$, $f(x_3) = 16$
 Lagrange's formula is

$$\begin{aligned}
 f(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) \\
 &+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1) \\
 &+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) \\
 &+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3) \\
 &+ \frac{(x - 5)(x - 9)(x - 11)}{(6 - 5)(6 - 9)(6 - 11)} (13) \\
 &+ \frac{(x - 5)(x - 6)(x - 11)}{(9 - 5)(9 - 6)(9 - 11)} (14) \\
 &+ \frac{(x - 5)(x - 6)(x - 9)}{(11 - 5)(11 - 6)(11 - 9)(5 - 9)(5 - 11)} (12) \\
 &+ \frac{7}{12} (x - 5)(x - 6)(x - 11) \\
 &+ \frac{4}{15} (x - 5)(x - 6)(x - 9) \\
 &= -\frac{1}{2} (x - 6)(x - 9)(x - 11) + \frac{13}{15} (x - 5)(x - 9)(x - 11)
 \end{aligned}$$

Putting $x = 10$, we get

$$\begin{aligned}
 f(10) &= -\frac{1}{2} (10 - 6)(10 - 9)(10 - 11) + \frac{13}{15} (10 - 5)(10 - 9)(10 - 11) \\
 &\quad - \frac{7}{12} (10 - 5)(10 - 6)(10 - 11) + \frac{4}{15} (10 - 5)(10 - 6)(10 - 9) \\
 &= 14.66666667
 \end{aligned}$$

Hence,

$$y(10) = 14.66666667$$

Example 2. Compute the value of $f(x)$ for $x = 2.5$ from the following table:

$x:$	1	2	3	4
$f(x):$	1	8	27	64

using Lagrange's interpolation method.

Solution. Here $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$

$$f(x_0) = 1, f(x_1) = 8, f(x_2) = 27, f(x_3) = 64$$

Lagrange's formula is

$$\begin{aligned}
 f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) \\
 &+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \\
 &= \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} (1) + \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} (8) \\
 &+ \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} (27) \\
 &+ \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} (64) \\
 &= -\frac{1}{6} (x-2)(x-3)(x-4) + 4(x-1)(x-3)(x-4) \\
 &- \frac{27}{2} (x-1)(x-2)(x-4) \\
 &+ \frac{32}{3} (x-1)(x-2)(x-3)
 \end{aligned}$$

Given $x = 2.5$, we get

$$\begin{aligned}
 f(2.5) &= -\frac{1}{6} (2.5-2)(2.5-3)(2.5-4) \\
 &\quad + 4(2.5-1)(2.5-3)(2.5-4) \\
 &\quad - \frac{27}{2} (2.5-1)(2.5-2)(2.5-4) \\
 &\quad + \frac{32}{3} (2.5-1)(2.5-2)(2.5-3)
 \end{aligned}$$

$$\begin{aligned}
& -\frac{27}{2}(2.5-1)(2.5-2)(2.5-4) \\
& +\frac{32}{3}(2.5-1)(2.5-2)(2.5-3) \\
& = 15.625
\end{aligned}$$

Hence, $f(2.5) = 15.625$.

Example 3. Find the cubic Lagrange's interpolating polynomial from the following data:

$x:$	0	1	2	5
$f(x):$	2	3	12	147.

Solution. Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

$$f(x_0) = 2, f(x_1) = 3, f(x_2) = 12, f(x_3) = 147$$

Lagrange's formula is

$$\begin{aligned}
f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) \\
&+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\
&+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) \\
&+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \\
&+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} (12) \\
&+ \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} (147) \\
&= (0-1)(0-2)(0-5)(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} (3) \\
&+ \frac{49}{20} x(x-1)(x-2) \\
&= -\frac{1}{5}(x^3 - 8x^2 + 17x - 10) + \frac{3}{4}(x^3 - 7x^2 + 10x) - 2(x^3 - 6x^2 + 5x) \\
&\quad + \frac{49}{20}(x^3 - 3x^2 + 2x) \\
&\Rightarrow f(x) = x^3 + x^2 - x + 2
\end{aligned}$$

which is the required Lagrange's interpolating polynomial.

20.5 SUMMARY

In this unit, we have discussed the following concepts

- Lagrange's interpolation formula for unequal intervals
- Problems based on Lagrange's interpolation formula for unequal intervals

20.6 GLOSSARY

Unequal Intervals: Data points whose independent-variable intervals are not constant.

Lagrange Interpolation: An interpolation method that does not require equally spaced data.

Lagrange Polynomial: A polynomial constructed from interpolation basis polynomials.

Basis Polynomial: A polynomial associated with one interpolation data point.

Divided Difference: A difference quotient used in interpolation with unequal intervals.

Newton's Divided Difference Formula: An interpolation formula based on divided differences.

Interpolation Nodes: The known data points used to construct an interpolation polynomial.

Interpolating Polynomial: A polynomial passing through the specified data points.

20.7 EXERCISES

1. Apply Lagrange's formula to find $f(5)$ and $f(6)$ given that

$$f(2) = 4, f(1) = 2, f(3) = 8, f(7) = 128$$

Explain why the result differs from those obtained by completing the series of powers of 2 ?

2. Values of $f(x)$ for values of x are given as

$$f(1) = 4, f(2) = 5, f(7) = 5, f(8) = 4$$

Find $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum.

3. Find by Lagrange's formula, the value of

(i) u_5 if $u_0 = 1, u_3 = 19, u_4 = 49, u_6 = 181$

(ii) u_4 if $u_3 = 16, u_5 = 36, u_7 = 64, u_8 = 81$ and $u_9 = 100$.

4. Using Lagrange's formula, find the values of

(i) y_5 if $y_1 = 4, y_3 = 120, y_4 = 340, y_5 = 2544$

(ii) y_0 if $y_{-30} = 30, y_{-12} = 34, y_3 = 38, y_{18} = 42$.

20.8 SUGGESTED READINGS

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